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Equivalent Quasi-Convex Form of the Multicast Max–Min Beamforming Problem

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***Abstract*—Multicast downlink transmission in a multi-cell network with multiple users is investigated. Max–min beamforming (MB) enables a fair distribution of the signal-to-interference-plus-noise-ratio among all users in a network for given power constraints at the base stations of the network. The multicast MB problem (MBP) is proved to be $\mathcal{NP}$-hard and non-convex in general. However, the MBP has an equivalent quasi-convex form and can be optimally solved with an efficient algorithm for special instances, depending on the structure of the available channel state information (CSI). This paper derives the equivalent quasi-convex form of the MBP for the practically relevant scenario of long-term CSI in the form of Hermitian positive semi-definite Toeplitz matrices and per-antenna array power constraints. The optimization problem is then given by a convex feasibility check problem with finite auto-correlation sequences (FASs) as optimization variables. Using FASs the MBP can be expressed as a quasi-convex fractional program. Based on the theory of quasi-convex programming, this paper proposes a fast root-finding algorithm with super-linear convergence.**

***Index Terms*—Multicast beamforming, max–min beamforming, spectral factorization, fractional programming**

I. Introduction

THIS paper regards a multicell network with multiple base stations (BSs) with the capability of beamforming. Each BS serves multiple users, each equipped with one antenna. In a given time slot, a BS transmits the same content to all users inside the cell which corresponds to the multicast scenario. If only one user per cell is scheduled, a unicast transmission is possible as well. The optimization of the beamforming vectors is based on available long-term channel state information (CSI). A global adaptation of the beam pattern can be achieved by closed loop multicast max–min beamforming (MB) based on the available CSI of all links in the considered network.

A. Related Work:

Multicell beamforming was intensively investigated since 1998. First works [1], [2] regard the unicast beamforming problem where each user gets different content. The multi-cell unicast max–min beamforming problem (MBP) has an equivalent quasi-convex (QC) form and recent works have presented an optimal low complexity algorithm based on uplink–downlink duality [3], [4], [5]. The multicast beamforming problem (BP) contains the unicast BP as a special case where all multicast groups contain exactly one user. The multicast BP has no equivalent dual uplink problem and is $\mathcal{NP}$-hard in general.

In 2004, Sidropoulos et al. [6] proposed the multicast power minimization problem (PMP) defined by a minimization of the total transmit power given a minimal signal-to-noise ratio (SNR). The authors further introduce the generalization of PMP called multicast MBP defined by the maximization of the minimum SNR given a total power budget. These so-called single group multicast beamforming problems are proved to be $\mathcal{NP}$-hard in general by the same authors in [7]. The work [8] extends the work [7] to the case of multiple multicast groups with interference among different groups in a single cell. The authors propose a semi-definite relaxation technique to find local optimal solutions.

Recent works on multicast beamforming regard approximations of the optimal solutions of the multicast BP [9], [10], [11], [12]. These works find near optimal solutions with less complexity than convex solver based methods. Although the multicast BPs are $\mathcal{NP}$-hard and non-convex in general, QC or convex equivalent forms for special instances may exist. A first case of the multicast MBP with an equivalent QC form was presented in 2007 by Karipidis et al. [8]. The authors derived an equivalent convex form for the PMP and an equivalent QC form of the MBP if the BS uses uniform linear arrays (ULAs) and far-field line-of-sight propagation conditions are given.

B. Contribution:

In multi-cell networks, instead of instantaneous CSI, the usage of long-term CSI in the form of higher rank spatial correlation matrices is more realistic. Therefore, this paper regards the multicast MBP based on long-term CSI with per-antenna array power constraints.

- This paper extends the work of [8] and proves the existence of a rank-1 solution of the relaxed semi-definite feasibility check problem of the MBP in the case of long-term CSI in the form of Hermitian positive semi-definite Toeplitz (HPST) matrices. This paper regards a multicell scenario where a joint optimization of all beamforming vectors given per-antenna array power constraints is presented.

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- Furthermore, this paper presents the equivalent QC form for the multicast MBP for long-term CSI in the form of HPST matrices and per-antenna array power constraints. This QC form uses FASs as optimization variables and the optimal beamforming vectors are recovered by spectral factorization.
- The equivalent QC form can be solved with a simple bisection based algorithm with linear convergence. Besides this standard solution, this paper proposes an algorithm with super-linear convergence based on a so-called parametric program of the equivalent QC fractional program of the MBP.

II. System Setup and Data Model

The most important notations are given in Table I below.

TABLE I: Overview of the notations.

a	scalar
a^*	conjugate complex
$\mathbf{a}$	vector
$\mathbf{A}$	matrix
$[\mathbf{a}]_n$	nth element of a vector
$[\mathbf{A}]_{n,m}$	element with indices n, m of a matrix $\mathbf{A}$
$[\mathbf{A}]_{:,m}$	mth column vector of a matrix $\mathbf{A}$
$\mathbf{A}^H$	conjugate transpose of a matrix $\mathbf{A}$
$\mathbf{I}_N$	identity matrix of dimension $N \times N$
$\mathbf{0}_N$	zero matrix of dimension $N \times N$
$\succcurlyeq$	matrix inequality on a cone of non-negative definite matrices
$\mathrm{Tr}(\mathbf{A})$	trace operation on a square matrix $\mathbf{A}$

A multiuser multicell network is considered with a set $\mathcal{S}$ of N_C cooperating BSs equipped with N_A antennas each, serving a set $\mathcal{U}$ of M users, each equipped with a single antenna. A group of users is served by one BS $c(i)$, e.g., $N_C \leq M$. The signal r_i received at a time instant by a user i is given by

$$r_i = \mathbf{h}_{c(i),i}^H \boldsymbol{\omega}_{c(i)} s_{c(i)} + \sum_{c \in \mathcal{S},\, c \neq c(i)} \mathbf{h}_{c,i}^H \boldsymbol{\omega}_c s_c + n_i, \quad (1)$$

where $\mathbf{h}_{c,i} \in \mathbb{C}^{N_A \times 1}$ is the channel vector from the cth BS to the ith user. The transmit beamforming vector at BS c is $\boldsymbol{\omega}_c = [\omega_c(0), \ldots, \omega_c(N_A - 1)]^T \in \mathbb{C}^{N_A \times 1}$, s_c is the information symbol transmitted by BS c. The symbols have unit power $\mathbb{E}\{|s_c|^2\} = 1$ and are uncorrelated $\mathbb{E}\{s_c s_k^*\} = 0$ if $c \neq k$. The variable $n_i \sim \mathcal{CN}(0, \sigma_i^2)$ is the complex additive Gaussian noise with $\mathbb{E}\{n_i\} = 0$ and variance $\mathbb{E}\{|n_i|^2\} = \sigma_i^2$. The beamforming matrix is given by $\boldsymbol{\Omega} = [\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_{N_C}]$.

Instead of instantaneous SINR $\hat{\gamma}_i(\boldsymbol{\Omega})$, long-term CSI is often used in a multi-cell optimization due to its longer stationary interval and, therefore, reduced required CSI feedback rate. The approximation of the ergodic capacity $\hat{R}_i = \mathbb{E}\{\log(1 + \hat{\gamma}_i(\boldsymbol{\Omega}))\} \approx \log(1 + \gamma_i^D(\boldsymbol{\Omega})) = R_i$ [13] results in the mean SINR

$$\gamma_i^D(\boldsymbol{\Omega}) = \frac{\boldsymbol{\omega}_{c(i)}^H \mathbf{R}_{c(i),i} \boldsymbol{\omega}_{c(i)}}{\sum_{\substack{c \in \mathcal{S} \\ c \neq c(i)}} \boldsymbol{\omega}_c^H \mathbf{R}_{c,i} \boldsymbol{\omega}_c + 1} \quad (2)$$

which is an often used metric in multicell beamforming [2]. Here, an additional expectation over the channel realizations $\mathcal{H}$ is made. The result is the spatial correlation matrix given by

$$\mathbf{R}_{c,i} = \frac{1}{\sigma_i^2} \mathbb{E}_{\mathcal{H}}\{\mathbf{h}_{c,i} \mathbf{h}_{c,i}^H\}.$$

Note the spatial correlation matrices are normalized by the noise power. Considering a ULA with N_A antenna elements and an antenna spacing δ ($d = \delta/\lambda$, where λ is the wave length) at the BS, the spatial correlation matrix is given by a Toeplitz matrix and can be decomposed to $\mathbf{R}_{c,i} = \mathbf{A}(\boldsymbol{\theta}_{c,i}) \mathbf{P}_{c,i} \mathbf{A}(\boldsymbol{\theta}_{c,i})^H$ [14]. Using this notation, the spatial correlation is observed as a combination of N_P uncorrelated waves with directions of arrival given by $\boldsymbol{\theta}_{c,i} = [\theta_{c,i,1}, \ldots, \theta_{c,i,N_P}]$ and path powers given by $\mathbf{P}_{c,i} = \mathrm{diag}(q_{c,i,1}, \ldots, q_{c,i,N_P}) \in \mathbb{R}^{N_P \times N_P}$. The matrix

$$\mathbf{A}(\boldsymbol{\theta}_{c,i}) = [\mathbf{a}(\theta_{c,i,1}), \ldots, \mathbf{a}(\theta_{c,i,N_p})] \in \mathbb{C}^{N_A \times N_p}$$

is a Vandermonde matrix containing the steering vectors: $\mathbf{a}(\theta_{c,i,p}) = [1, \exp(\zeta \sin(\theta_{c,i,p})), \ldots, \exp(\zeta(N_A - 1)\sin(\theta_{c,i,p}))]^T$, $\zeta = j2\pi d$. The formulation of the long-term CSI can be rewritten as:

$$\mathbf{R}_{c,i} = \sum_{p=1}^{N_P} q_{c,i,p} \mathbf{a}(\theta_{c,i,p}) \mathbf{a}(\theta_{c,i,p})^H. \quad (3)$$

The matrix in (3) is a HPST matrix which is a reasonable assumption if ULAs are used at the BSs [15], [16]. In practice, there exists different methods to approximate the Toeplitz matrices based on finite noise sequences, e.g., the method in [17].

In a multicell scenario a sum power constraint as in, e.g., [8] is not a practically relevant assumption. Then per-BS antenna array power constraints are practically more relevant. In this case, each antenna array c of a BS will be subject to a total power budget P_c. The convex cone of beamforming vectors fulfilling the per-BS antenna array power constraints is given by

$$\mathcal{P} = \{\boldsymbol{\Omega} \in \mathbb{C}^{N_A \times N_C} : \boldsymbol{\omega}_c = [\boldsymbol{\Omega}]_{:,c},\ \boldsymbol{\omega}_c^H \boldsymbol{\omega}_c \leq P_c \ \ \forall c \in \mathcal{S}\}. \quad (4)$$

III. Optimization Problem

It is desired to improve the worst SINR of the currently scheduled users. Therefore, the MBP can be stated as

$$\gamma^D = \max_{\boldsymbol{\Omega} \in \mathcal{P}} \ \min_{i \in \mathcal{U}} \gamma_i^D(\boldsymbol{\Omega}). \quad (5)$$

A balanced SINR can be the result. The problem (5) is non-convex in general because of the non-convex objective function $f(\boldsymbol{\Omega}) = \min_{i \in \mathcal{U}} \ \gamma_i^D(\boldsymbol{\Omega})$. However, this problem can be relaxed to a QC problem. It is desired to maximize $f(\boldsymbol{\Omega})$, hence, the objective function must have an equivalent quasi-concave form to prove that the MBP has an equivalent QC form.

Definition 1: [18] A function $f(\boldsymbol{x})$ defined on a convex set $\mathcal{F}$ is quasi-concave if every upper level set $\mathcal{S}_\alpha = \{\boldsymbol{x} \in \mathcal{F} : f(\boldsymbol{x}) > \alpha\}$ of $f(\boldsymbol{x})$ is convex for every value of α.

The non-convex optimization problem (5) can always be relaxed to QC problem with a semi-definite program as a feasibility check problem. With semi-definite matrices $\mathbf{X}_c =$

$\omega_c \omega_c^H$ and $\mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_{N_C}]$, the downlink SINR is given by

$$\gamma_i^{\tilde{D}}(\mathbf{X}) = \frac{\mathrm{Tr}\{\mathbf{X}_{c(i)}\mathbf{R}_{c(i),i}\}}{\sum_{c\in\mathcal{S},c\neq c(i)} \mathrm{Tr}\{\mathbf{X}_c\mathbf{R}_{c,i}\} + 1}. \tag{6}$$

Removing the non-convex rank-1 constraint $\mathrm{rank}(\mathbf{X}_c) = 1 \ \forall c \in \mathcal{S}$, the MBP (5) with per-antenna array power constraints can be relaxed to

$$\gamma^{\tilde{D}} = \max_{\mathbf{X}} \ \min_{i\in\mathcal{U}} \gamma_i^{\tilde{D}}(\mathbf{X}) \tag{7}$$

$$\text{s.t.} \quad \mathrm{Tr}\{\mathbf{X}_c\} \leq P_c, \ \mathbf{X}_c \succeq 0 \ \ \forall c \in \mathcal{S}. \tag{8}$$

Let $\mathcal{F}$ be defined by (8). The constraints are convex and for a fixed γ, therefore, the upper level sets of the objective function given by $\mathcal{S}_\gamma = \{\mathbf{X} \in \mathcal{F} : \ \tilde{f}(\mathbf{X}) = \min_{i\in\mathcal{U}} \gamma_i^{\tilde{D}}(\mathbf{X}) > \gamma\}$ are convex, hence, problem (7) is QC. It can be solved by convex feasibility check problems in the form of semi-definite programs (SDPs) [18]. A bisection algorithm can iterate arbitrarily tightly to the value of the global optimum. This solution is a standard method of solving the MBP and is used as a reference in this paper. The work of Karipidis et al. proved that Problem (7) has rank-1 solution if instantaneous CSI in the form of Vandermonde channel vectors is given. In this paper, we proof the existence of a rank-1 solution in the case of long-term CSI in the form of HPST matrices, which is a practically more relevant assumption in multicell scenarios. We state this observation by the following proposition:

Proposition 1: In the case of long-term CSI in the form of matrices $\mathbf{R}_{c,i}$ given by (3), there exists a solution for the MBP (7) with all matrices $\mathbf{X}_c \ \forall c \in \mathcal{S}$ having rank-1.

Proof: The proof is presented in Appendix A. ■

Remarkably, a rank-1 solution also exists in the case of long-term CSI and ULAs at the BSs. Consequently, an optimal solution can be obtained in a multicell multicast scenario based on long-term CSI in the form of HSTP matrices. However, problem (7) is not guaranteed to always yield solutions with rank-1. The work of [8] describes the same observation for Vandermonde channels where also a rank-1 solution exists but the proposed semi-definite relaxation does not consistently yield rank-1 solutions. Therefore, the next part of this section proposes an equivalent QC form of the MBP (5). The derived solution yields quasi-optimal beamforming vectors based on an equivalent QC formulation of the original problem with FASs. The technique of convex optimization with FASs [19] to derive an equivalent QC form of the MBP as in [8] is extended in this paper to higher rank correlation matrices according to (3) with coefficients

$$\begin{aligned}
[\mathbf{R}_{c,i}]_{k,l} =& [\sum_{p=1}^{N_P} q_{c,i,p}\mathbf{a}(\theta_{c,i,p})\mathbf{a}(\theta_{c,i,p})^H]_{k,l} \\
=& \sum_{p=1}^{N_P} q_{c,i,p} e^{j2\pi d(k-l)\sin(\theta_{c,i,p})}.
\end{aligned} \tag{9}$$

With (9) and $n = k - l$, the coefficients of the HPST matrix are given by

$$r_{c,i}(n) = \sum_{p=1}^{N_P} q_{c,i,p} e^{j2\pi n \sin(\theta_{c,i,p})} \ \ \forall n = 0, \ldots, N_A - 1. \tag{10}$$

The idea of optimization with FASs is based on the following definitions and Lemma 1 of [19]: With $\tilde{\mathbf{E}}_k \in \{0,1\}^{N_A \times N_A}$ denoting the matrix which has zero entries except on the kth sub-diagonal where it has only ones, $k \in \{-N_A + 1, \ldots, 1, 0, 1, \ldots, N_A - 1\}$, observe that (3) can be rewritten as [20]

$$\mathbf{R}_{c,i} = \sum_{k=-N_A+1}^{N_A-1} r_{c,i}(k)\tilde{\mathbf{E}}_k. \tag{11}$$

A representation of variables by finite auto-correlation sequences results in convex or quasi-convex problems. With the shift matrix $\mathbf{E}^k$ which is the kth power of the matrix $\mathbf{E}$ which has zero entries except on the 1st lower sub-diagonal where it has only ones, the auto-correlation sequence is defined by

$$x(k) = \boldsymbol{u}^H \mathbf{E}^k \boldsymbol{u} = \mathrm{Tr}\{\mathbf{E}^k \boldsymbol{u}\boldsymbol{u}^H\}, \tag{12}$$

$k \in \{0, \ldots, K - 1\}$ with vectors $\boldsymbol{u} \in \mathbb{C}^{K\times 1}$. The matrix $\mathbf{E}^k$ has zeros everywhere, except on the k-th sub-diagonal. Equation (12) is non-convex because of the rank-1 constraint. Remarkably, the same set as in the definition of the FAS given by (12) can be described with the relaxed rank-1 constraint.

Lemma 1: [19] Using some positive semi-definite matrix $\mathbf{U} \succeq 0$ such that $\hat{x}(k) = \mathrm{Tr}\{\mathbf{E}^k\mathbf{U}\}$, $k = 0, ..., N$, $\hat{x}(k)$ describes the same set as $x(k) = \boldsymbol{u}^H\mathbf{E}^k\boldsymbol{u} = \mathrm{Tr}\{\mathbf{E}^k\boldsymbol{u}\boldsymbol{u}^H\}$.

Using Lemma 1, the non-convex MBP (5) can be expressed as a QC problem, where the optimization variables are converted to FASs. The reformulation leads to additional convex constraints. The optimal beamforming vectors can be obtained by spectral factorization techniques, e.g. [21]. In the work of Karipidis et al. [8], an equivalent QC form of the multicast MBP for the single cell Vandermonde channel scenario is presented. The Toeplitz property of the used long-term CSI also leads to an equivalent QC form of the multicast MBP. In this paper, we derive the QC form of the multicast MBP in the case of long-term CSI in the form of HPST matrices. We summarize the derivation in the following proposition:

Proposition 2: In the case of long-term CSI in the form of matrices $\mathbf{R}_{c,i}$ given by (11), and variables in the form of $\mathbf{x}_c = [x_c(0), \ldots, x_c(N_A - 1)]^T$, $\forall c \in \mathcal{S}$, semi-definite matrices $\mathbf{U}_c \succeq 0$ stored in $\boldsymbol{U} = [\mathbf{U}_1, \ldots, \mathbf{U}_{N_C}]$, the vectors $\mathbf{r}_{c,i} = [r_{c,i}(0), \ldots, r_{c,i}(N_A - 1)]$, and using the following matrix as in [8]

$$\tilde{\mathbf{I}}_{N_A} = \begin{bmatrix} 1 & \mathbf{0}_{1\times N_A-1} \\ \mathbf{0}_{N_A-1\times 1} & 2\mathbf{I}_{N_A-1} \end{bmatrix} \in \mathbb{N}^{N_A\times N_A}, \tag{13}$$

the equivalent QC form of the original problem (5) is given by

$$\max_{\gamma,\mathbf{x},\boldsymbol{U}} \quad \gamma \tag{14}$$

$$\text{s.t.} \quad \frac{\mathrm{Re}\{\mathbf{r}_{c(i),i}\tilde{\mathbf{I}}_{N_A}\mathbf{x}_{c(i)}\}}{\sum_{\substack{c\in\mathcal{S}\\ c\neq c(i)}} \mathrm{Re}\{\mathbf{r}_{c,i}\tilde{\mathbf{I}}_{N_A}\mathbf{x}_c\} + 1} \geq \gamma \ \ \forall i \in \mathcal{U}, \tag{15}$$

$$x_c(0) \leq P_c \quad \forall c \in \mathcal{S} \tag{16}$$

$$\begin{aligned}
& x_c(k) = \mathrm{Tr}\{\mathbf{E}^k\mathbf{U}_c\}, \ \ \mathbf{U}_c \succeq 0 \\
& \forall c \in \mathcal{S} \ \ \forall k = 0, \ldots, N_A - 1.
\end{aligned} \tag{17}$$

Proof: The proof is presented in Appendix B. ■

The Toeplitz property of long-term CSI in the case of a ULA offers, therefore, also a possibility to find optimal solutions for the multicast MBP. This new QC form is practically more relevant in multicell scenarios where per-BS antenna array power constraints are given and where only long-term CSI is available.

IV. Solution Based on Fractional Programming

In the previous section, a QC form of the multicast MBP is derived. This QC form can be solved with a simple bisection based algorithm. These algorithms have a linear convergence behavior. The QC form of the MBP can be also seen as a so-called generalized fractional program (FP). For these QC optimization problems, a super-linear converging algorithm is feasible. If long-term CSI in the form of HPST matrices is available, the equivalent QC form of the multicast MBP is given in Proposition 2. This QC form can be expressed as a QC FP:

Let $f_i : \mathbb{C}^N \to \mathbb{R}$ be a continuous and convex function and let $g_i : \mathbb{C}^N \to \mathbb{R}$ be a continuous and concave function on the convex set $\mathcal{X}$, consider the following QC problem:

$$\bar{\Theta} = \min_{\boldsymbol{x}\in\mathcal{X}} \max_{i\in\mathcal{I}} \frac{f_i(\boldsymbol{x})}{g_i(\boldsymbol{x})} \tag{18}$$

where $\mathcal{I}$ is a finite set of integers. This QC program can be solved with the following parametric program [22]:

$$F(\Theta) = \min_{\boldsymbol{x}\in\mathcal{X}} \max_{i\in\mathcal{I}} \{f_i(\boldsymbol{x}) - \Theta g_i(\boldsymbol{x})\}. \tag{19}$$

The idea of the algorithm presented in this section is based on searching the root of $F(\Theta) = 0$. If the parametric program (19) is very close to zero, the solution is near optimal and the solution of the parametric program is also a solution for the original generalized FP [22].

Proposition 3: The parametric program of the equivalent QC form (14) of the multicast MBP (5) is given by

$$\begin{aligned} F(\gamma) = \min_{\mathbf{x}\in\mathcal{X}} \max_{i\in\mathcal{U}} \{ & -\mathrm{Re}\{\mathbf{r}_{c(i),i}\tilde{\mathbf{I}}_{N_A}\mathbf{x}_{c(i)}\} \\ & -\gamma(\sum_{\substack{c\in\mathcal{S}\\ c\neq c(i)}} \mathrm{Re}\{\mathbf{r}_{c,i}\tilde{\mathbf{I}}_{N_A}\mathbf{x}_c\} + 1)\}. \end{aligned} \tag{20}$$

Proof: The proof is presented in Appendix C. ■

In [23], the authors propose a fast algorithm with super-linear convergence to solve a QC FP. The algorithm exploits results of [23, Theorem 2.1] which gives connections among the QC FP and its parametric program (20). According to [23, Theorem 2.1], if the parametric program (20) results in $F(\gamma) = 0$, the optimal SINR is found.

Proposition 4: If $F(\gamma) = 0$ a root finding Dinkelbach algorithm finds the optimal solution.

Proof: It is straightforward to show that $\mathcal{X}$ is compact. As already shown in [23], if $\mathcal{X}$ is compact and $F(\gamma) = 0$, then $\mathbf{x}$ achieves the optimal value. ■

As it can be observed from the numerical results, a root finding algorithm as in [22] always iterates to $F(\gamma) \approx 0$.

TABLE II: Simulation parameters.

Number of user drops	4000
Number of users per user drop	15
Number of BSs drop	3
Number of users per group drop	5
Transmit antenna arrays	ULA
Number of antenna array elements at BS	8
Number of antenna array elements at MS	1
Intersite distance	500 m
Antenna spacing	half wavelength
Path loss exponent	3.76
Available CSI	long-term CSI
Power angular density	Laplacian [24], 15°
Power constraint	per-BS

V. Numerical Results

Table II presents the main simulation parameters of the simulated multicell network. The scenario is assumed to be interference dominated. Two algorithms are compared in this section:

- A1: Conventional bisection method to solve problem (7).
- A2: Root finding algorithm with parametric program (20).

For the generation of the statistics, in total 4000 user drops are randomly generated. In each user drop, the long-term CSI in the form HPST matrices is generated based on the location of the users and BSs.

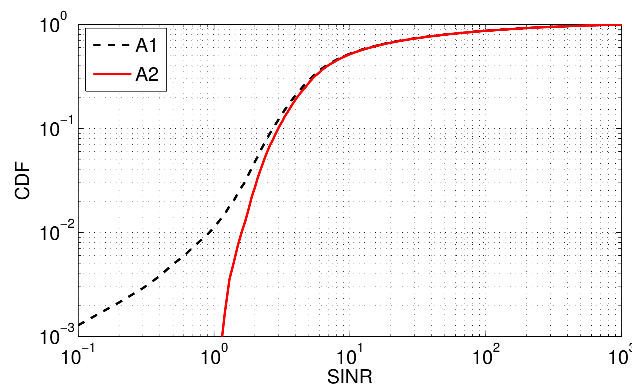

Fig. 1: Cumulative distribution function (CDF) of the SINR of the new algorithm based on the Dinkelbach iteration presented in Section IV (red) and the conventional SDP based bisection based method for the problem (7) (black).

Figure 1 shows the cumulative distribution function (CDF) of the SINR for a precision of $\epsilon = 10^{-5}$. Comparing both CDFs, the new algorithm A2 outperforms the conventional SDP based bisection A1 especially for the weakest users. The new algorithm A2 achieves a higher minimum SINR. The multicast beamforming problem (7), has a solution where all matrices $\mathbf{X}_c$ $\forall c \in \mathcal{S}$ have rank 1. However, problem (7) is not guaranteed to always yield solutions with rank 1 [8]. There are cases where the feasibility check problem of (7) has higher rank solutions. Figure 2 presents the CDF of the number of iterations algorithms A1 and A2 required for the given precision $\epsilon = 10^{-5}$. As it can be observed from this figure, A2 requires less iterations than A1 due to its super-linear convergence in multiple cases (95%). However, in a few

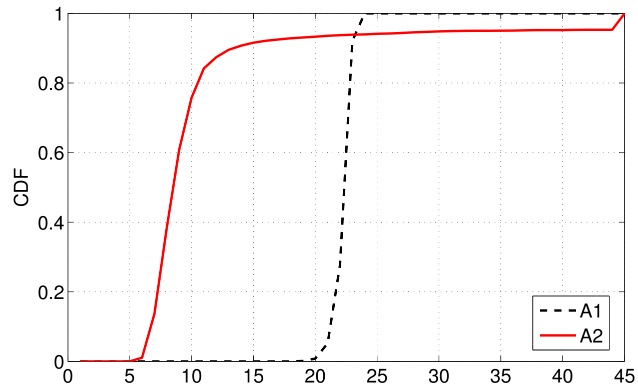

Fig. 2: Cumulative distribution function (CDF) of the number of iterations for a precision ϵ.

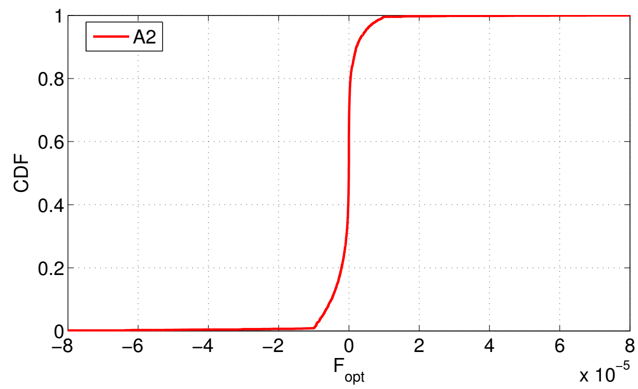

Fig. 3: Cumulative distribution function (CDF) of the solution of the parametric program.

cases (5%), the new algorithm converges more slowly. In these cases, the algorithm aborts after 45 iterations and takes the solution of the last iteration. Finally, Figure 3 shows the value of $F(\gamma)$ of the parametric program. As it can be observed from this figure, the solution of the parametric program is nearly optimal and in more than 95% percent of the simulation runs $F(\gamma)$ is smaller than the precision interval of $\epsilon = 10^{-5}$.

VI. Conclusions

This paper investigates the multicast max–min beamforming problem with per-antenna array power constraints. The max–min beamforming problem can be solved with a convex feasibility check problem with linear convergence in the form of a semi-definite program. This paper proves the existence of a rank-1 solution if long-term CSI in the form of positive semi-definite Hermitian Toeplitz matrices is available.

Furthermore, this paper proposes an equivalent problem based on convex programming with finite autocorrelation sequences. The resulting equivalent problem is a QC FP which can be solved by a root finding algorithm. The new method is also based on a semi-definite program, however, it finds optimal solution values and has super-linear instead of merely linear convergence. A future work could be an extension to large multicast groups consisting of multiple BSs.

Appendix A
Proof of Proposition 1

An optimal solution $\mathbf{X}_c$ of the SDP is assumed. In general, this solution has rank $\rho_c = \text{rank}(\mathbf{X}_c)$ larger than one. Thus $\mathbf{X}_c$ can be decomposed to $\mathbf{X}_c = \sum_{r=1}^{\rho_c} \hat{\boldsymbol{\omega}}_{c,r}\hat{\boldsymbol{\omega}}_{c,r}^H$. Next it is shown that a rank-1 solution $\mathbf{\Psi}_c = \boldsymbol{\omega}_c\boldsymbol{\omega}_c^H$ can achieve the optimum. The signal power received at a user i from BS array c is given by

$$\begin{aligned}\text{Tr}(\mathbf{X}_c\mathbf{R}_{c,i}) &= \text{Tr}(\sum_{r=1}^{\rho_c} \hat{\boldsymbol{\omega}}_{c,r}\hat{\boldsymbol{\omega}}_{c,r}^H\mathbf{R}_{c,i})\\ &= \sum_{r=1}^{\rho_c} \text{Tr}(\hat{\boldsymbol{\omega}}_{c,r}\hat{\boldsymbol{\omega}}_{c,r}^H\mathbf{R}_{c,i}).\end{aligned}$$

Using the decomposition of positive semi-definite Toeplitz matrices according to (3), the received signal can be simplified to:

$$\begin{aligned}\text{Tr}(\mathbf{X}_c\mathbf{R}_{c,i}) &= \sum_{r=1}^{\rho_c} \text{Tr}(\hat{\boldsymbol{\omega}}_{c,r}\hat{\boldsymbol{\omega}}_{c,r}^H \sum_{p=1}^{N_P} q_{c,i,p}\mathbf{a}(\theta_{c,i,p})\mathbf{a}(\theta_{c,i,p})^H)\\ &= \sum_{p=1}^{N_P} q_{c,i,p} \sum_{r=1}^{\rho_c} |\mathbf{a}(\theta_{c,i,p})^H\hat{\boldsymbol{\omega}}_{c,r}|^2. \qquad (21)\end{aligned}$$

Now the proof follows the same idea as in [8]: the non-negative complex trigonometric polynomial $\sum_{r=1}^{\rho_c} |\mathbf{a}(\theta_{c,i,p})^H\hat{\boldsymbol{\omega}}_{c,r}|^2 \geq 0$ is positive for any value of $\theta_{c,i,p} \in [0, 2\pi)$. the Riesz-Féjer theorem [25], we can find a vector $\boldsymbol{\omega}_c$ which does not dependent on $\theta_{c,i,p}$ such that for all $\theta_{c,i,p}$ [8] $\sum_{r=1}^{\rho_c} |\mathbf{a}(\theta_{c,i,p})^H\hat{\boldsymbol{\omega}}_{c,r}|^2 = |\mathbf{a}(\theta_{c,i,p})^H\boldsymbol{\omega}_c|^2$ holds. Inserting this in (21) results in

$$\begin{aligned}\text{Tr}(\mathbf{X}_c\mathbf{R}_{c,i}) &= \sum_{p=1}^{N_P} q_{c,i,p}|\mathbf{a}(\theta_{c,i,p})^H\boldsymbol{\omega}_c|^2\\ &= \text{Tr}(\boldsymbol{\omega}_c\boldsymbol{\omega}_c^H \sum_{p=1}^{N_P} q_{c,i,p}\mathbf{a}(\theta_{c,i,p})\mathbf{a}(\theta_{c,i,p})^H)\\ &= \text{Tr}(\boldsymbol{\omega}_c\boldsymbol{\omega}_c^H\mathbf{R}_{c,i}).\end{aligned}$$

Thus, an equivalent rank-1 positive semi-definite matrix $\mathbf{X}_c = \boldsymbol{\omega}_c\boldsymbol{\omega}_c^H$ exists, which results in the same receive power at the user i, consequently, the convex feasibility check problem of the MBP (7) has a solution $\mathbf{X}$ with all matrices $\mathbf{X}_c$ $\forall c \in \mathcal{S}$ having rank 1.

Appendix B
Proof of Proposition 2

The objective function $f(\mathbf{\Omega})$ can be proved to have an equivalent quasi-concave form if the upper level sets of the objective function are convex. With (11) as in [20], the signal power received at a user i by the ULA c is given by

$$\begin{aligned}\boldsymbol{\omega}_c^H \mathbf{R}_{c,i} \boldsymbol{\omega}_c =& \boldsymbol{\omega}_c^H (\sum_{k=-N_A+1}^{N_A-1} r_{c,i}(k) \tilde{\mathbf{E}}_k) \boldsymbol{\omega}_c \\ =& \sum_{k=-N_A+1}^{N_A-1} r_{c,i}(k) \boldsymbol{\omega}_c^H \tilde{\mathbf{E}}_k \boldsymbol{\omega}_c \\ =& \sum_{k=-N_A+1}^{N_A-1} r_{c,i}(k) x_c(k). \end{aligned} \quad (22)$$

Furthermore, $x_c(k) = \boldsymbol{\omega}_c^H \tilde{\mathbf{E}}_k \boldsymbol{\omega}_c = \mathrm{Tr}\{\tilde{\mathbf{E}}_k \boldsymbol{\omega}_c \boldsymbol{\omega}_c^H\}$ is an FAS [19], consequently, (22) is a linear function over $x_c(k) \in \mathbb{C}$ with the coefficients $r_{c,i}(k)$. It is evident that the sequences are conjugate symmetric $r_{c,i}(-k) = r_{c,i}(k)^*,\ x_c(-k) = x_c(k)^*$. Therefore, (22) can be rewritten as in [8] as:

$$\boldsymbol{\omega}_c^H \mathbf{R}_{c,i} \boldsymbol{\omega}_c = x_c(0) r_{c,i}(0) + 2 \sum_{k=1}^{N_A-1} \mathrm{Re}\{x_c(k) r_{c,i}(k)\}. \quad (23)$$

The received power (23) can be simplified to $\boldsymbol{\omega}_c^H \mathbf{R}_{c,i} \boldsymbol{\omega}_c = \mathrm{Re}\{\mathbf{r}_{c,i} \tilde{\mathbf{I}}_{N_A} \mathbf{x}_c\}$. Besides the received power, also the per-BS antenna array power constraints can be rewritten with the use of FASs $\boldsymbol{\omega}_c^H \boldsymbol{\omega}_c = \boldsymbol{\omega}_c^H \mathbf{E}^0 \boldsymbol{\omega}_c = x_c(0)$. With $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{N_C}]$ and the set of positive semi-definite auxiliary matrices $\{\mathbf{U}_c,\ \forall c \in \mathcal{S}\}$, with $\mathbf{U}_c \in \mathbb{C}^{N_A \times N_A}$, the set

$$\begin{aligned}\mathcal{X} = \{\mathbf{x} : x_c(k) = \mathrm{Tr}\{\mathbf{E}^k \mathbf{U}_c\},\ x_c(0) \le P_c, \\ \mathbf{U}_c \succeq 0\ \ \forall c \in \mathcal{S}\ \ \forall k = 0, \ldots, N_A - 1\}\end{aligned} \quad (24)$$

is convex. Due to Lemma 1, the constraint $x_c(k) = \mathrm{Tr}\{\mathbf{E}^k \mathbf{U}_c\}$ describes the same set as $x_c(k) = \mathrm{Tr}\{\tilde{\mathbf{E}}_k \boldsymbol{\omega}_c \boldsymbol{\omega}_c^H\}$. Due to convexity of $\mathcal{X}$, the upper level sets

$$\mathcal{S}_{\gamma,i} = \{\mathbf{x} \in \mathcal{X} :\ \frac{\mathrm{Re}\{\mathbf{r}_{c(i),i} \tilde{\mathbf{I}}_{N_A} \mathbf{x}_c\}}{\sum_{\substack{c \in \mathcal{S} \\ c \neq c(i)}} \mathrm{Re}\{\mathbf{r}_{c,i} \tilde{\mathbf{I}}_{N_A} \mathbf{x}_c\} + 1} \ge \gamma\} \quad (25)$$

are also convex. The constraints (15)-(17) are convex constraints [19]. For a fixed γ, the final problem (14)-(17) is equivalent to (5) and is an SDP [8], which is known to be convex. Hence, the original problem (5) has an equivalent QC form in the case of long-term CSI in the form of HPST matrices.

Appendix C
Proof of Propostion 3

The proof is straightforward. With the negative affine functions

$$f_i(\mathbf{x}) = -\,\mathrm{Re}\{\mathbf{r}_{c(i),i} \tilde{\mathbf{I}}_{N_A} \mathbf{x}_{c(i)}\}$$

and the positive affine functions

$$g_i(\mathbf{x}) = \sum_{\substack{c \in \mathcal{S} \\ c \neq c(i)}} \mathrm{Re}\{\mathbf{r}_{c,i} \tilde{\mathbf{I}}_{N_A} \mathbf{x}_c\} + 1,$$

the equivalent FP is $\bar{\gamma} = -\min_{\mathbf{x} \in \mathcal{X}}\ \max_{i \in \mathcal{U}} \frac{f_i(\mathbf{x})}{g_i(\mathbf{x})}$ which is equivalent to (14)-(17).

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