

Personal use of this material is permitted. Permission from IEEE must be obtained for all other uses, in any current or future media, including reprinting/republishing this material for advertising or promotional purposes, creating new collective works, for resale or redistribution to servers or lists, or reuse of any copyrighted component of this work in other works.

On the Duality of the Max–Min Beamforming Problem with Per-Antenna and Per-Antenna Array Power Constraints

Guido Dartmann, *Student Member, IEEE*, Xitao Gong, *Student Member, IEEE*, Waqas Afzal, Gerd Ascheid, *Senior Member, IEEE*

Abstract—This paper considers a downlink unicast transmission in a multi-cell network with multiple users. In a network with frequency reuse-1, inter-cell interference is a limiting factor. The max–min beamforming (MB) technique enables a balancing of the signal-to-interference-plus-noise ratio among all users in a network under a power budget. Thus, a fair distribution of the achievable rate can be achieved. The MB problem (MBP) is non-convex in general. However, if instantaneous channel state information is available, the MBP has an equivalent quasi-convex form and can be optimally solved with an efficient algorithm based on a convex solver.

In addition to this convex solver based solution, this paper considers a so-called surrogate dual problem of the MBP with, per-antenna and per-antenna array power constraints. The surrogate dual problem combines multiple power constraints to a single power constraint. Furthermore, the surrogate dual problem can be efficiently solved for long-term channel state information (CSI) in the form of spatial correlation matrices. Strong duality is proved for instantaneous CSI and long-term CSI in the form of higher rank spatial correlation matrices. With the surrogate dual problem, a fast algorithm for the MBP is presented. The convergence issue is discussed. Numerical results verify the convergence and the performance of the proposed algorithm.

Index Terms—Multiuser, max–min beamforming, quasi convex, duality, surrogate dual problem, long-term CSI

I. INTRODUCTION

IN a multi-cell wireless network with frequency reuse-1, interference (intra-cell and inter-cell) is a limiting factor. Especially users far away from their serving base station (BS) are subject to strong interference of adjacent BSs. The result is an unfair distribution of the signal-to-interference-plus-noise-ratio (SINR) among the users in the network. The use of smart antennas at the BSs allows mitigation of interference and, therefore, the improvement of the SINR of cell edge users. This can be achieved by closed loop beamforming, i.e.,

Parts of this work were presented at the 74th IEEE Vehicular Technology Conference (VTC-Fall), San Francisco (CA) 2011. Copyright ©2012 IEEE. Personal use of this material is permitted. However, permission to use this material for any other purposes must be obtained from the IEEE by sending a request to pubs-permissions@ieee.org. Guido Dartmann, Xitao Gong and Gerd Ascheid are with the Chair for Integrated Signal Processing Systems, RWTH Aachen University, Templergraben 55, 52056 Aachen, Germany; tel.: +49 241 80 {27871, 27885, 27883}; fax: +49 241 80 22195; email: {dartmann, gong, ascheid}@iss.rwth-aachen.de. Waqas Afzal was with the Chair for Integrated Signal Processing Systems, RWTH Aachen University, Templergraben 55, 52056 Aachen, Germany; email: waqas.afzal@rwth-aachen.de and is now with the King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia

by an optimization of the weighting factors of each antenna element based on the available channel state information (CSI) of all links in the considered network. This paper considers two different cases of generally available CSI:

- 1) Instantaneous CSI requires a fast CSI feedback mechanism. Especially in a large network with multiple BSs, a fast backbone network based on fiber is essential to update the CSI of all users in the network.
- 2) Long-term CSI in the form of spatial correlation matrices requires an update in the order of the length of each local stationary interval, which is less frequent compared to the use of instantaneous CSI.

A promising optimization technique to improve the fairness based on CSI is called max–min beamforming (MB). The MB problem (MBP) is non-convex in general [1]. However, in the case of instantaneous CSI an equivalent quasi-convex form is known. In this context, a quasi-convex optimization means that the maximization problem has a quasi-concave objective function and convex constraints. Therefore, a global optimal value exists and, e.g., a bisection algorithm can converge arbitrarily closely to this optimal value if the bisection interval is correctly chosen [2]. In [3], a equivalent quasi-convex form of the unicast MBP is derived for instantaneous CSI and per-antenna power constraints. If a sum power constraint is given, the problem is proved to be optimally solvable for both, instantaneous CSI or arbitrary long-term CSI and a sum power constraint [4]. An open question is whether the MBP can be solved optimally in the case of long-term CSI given by higher rank spatial correlation matrices and subject to more general power constraints.

Related work: Transmit beamforming has been investigated intensively since 1998. A central aspect of this field is the uplink–downlink duality in unicast beamforming. First works regarding this aspect are [5], [6]. They regard the power minimization problem (PMP) as well as the MBP. Various publications generalize their work or give further proofs and extensions. An overview of these publications is given in Tab. I. A deep information theoretic background concerning the duality of the Gaussian broadcast channel is studied in [7].

Unicast beamforming is non-convex with general power constraints and if long-term CSI is used [1]. However, several equivalent quasi-convex or even convex forms are currently identified for some special cases. Table II gives an overview of all related problems.

Based on long-term CSI, the unicast beamforming problem

of minimization of the sum power subject to individual SINR constraints was investigated in [1]. This problem is called PMP. In [1], the authors showed that the non-convex PMP can be relaxed to a convex semidefinite program which has no relaxation gap. The unicast PMP and the unicast MBP are reciprocal problems [8]. In [4], the authors presented an optimal solution for the MBP based on long-term CSI and a sum power constraint. The convergence of the iterative low complexity algorithm is later proved in [9].

The framework of the Lagrangian duality theory was introduced to solve the unicast multiuser beamforming problem in the work of [8] and [12]. In the case of instantaneous CSI, the PMP has an equivalent convex form, as opposed to the MBP which has a merely quasi-convex form. The authors in [12] propose a fast direct solution for the minimization of the maximum power margin over all antennas subject to per-antenna power constraints and individual SINR constraints. The authors of [19] extended the work to the multi-cell case. The PMP requires a properly chosen SINR. The MBP is more flexible and overcomes this complication. The MBP allows finding the highest SINR constraint such that the problem is feasible. In [15], the authors investigated the max–min weighted SINR problem with individual power constraints. The authors derived a fast algorithm based on the non-linear Perron Frobenius theory. In [3], the authors presented an equivalent quasi-convex form of the unicast MBP with per-BS antenna array power constraints for the case of instantaneous CSI. Similar to [8], the authors presented an approach in the form of a feasibility check problem given as second order cone problem (SOCP). The work [20] presented a fast direct solution for the MBP with long-term CSI and per-BS antenna array power constraints based on the framework of the Lagrangian duality theory. This scenario is not proved to have an equivalent convex form, hence, strong duality is not given in general. In [21], the MBP based on instantaneous CSI was investigated and the solution is extended to per-antenna power constraints.

Contributions: This paper considers the duality of the MBP with per-antenna or per-antenna array power constraints.

- This paper presents an approach involving an equivalent dual uplink problem for the MBP with multiple power constraints based on the surrogate dual problem. If instantaneous CSI is available, an equivalent quasi-convex form of the MBP exists and strong surrogate duality can be directly proved with a duality theorem for quasi-convex programming [22]. The surrogate dual problem is proved to be quasi-convex, hence, it can be solved efficiently.
- If only long-term CSI in the form of higher rank spatial correlation matrices is available no equivalent quasi-convex form is known and the surrogate duality theorem of [22] can not be applied. An additional proof is presented to show strong duality also in this more general case.
- The surrogate dual problem of the MBP can be proved to be equal to the Lagrangian dual problem of the MBP and, at the optimum, the combined power constraint of the surrogate dual problem is satisfied with equality.
- The surrogate dual problem of the MBP leads to an

iterative low complexity algorithm. The convergence of the proposed algorithm is discussed.

Outline: Section II gives a detailed description of the system model for unicast downlink beamforming. Section III introduces the MBP. Furthermore, this section derives an equivalent quasi-convex form of the MBP with multiple power constraints given that instantaneous CSI is available (Section III-B). Finally, Subsection III-C presents the surrogate dual problem of the primal downlink MBP and gives a proof for strong duality. Section IV presents a fast algorithm for the unicast MBP with per-antenna power constraints based on the surrogate dual problem. Finally, the paper underlines the performance of the presented iterative solution in Section V with some numerical results and concludes with a summary in Section VI.

Notation: Lower case and upper case boldface symbols denote vectors and matrices, respectively. The i th element of a vector is denoted by $[\mathbf{a}]_i$. The element indexed n, m of a matrix \mathbf{A} is denoted by $[\mathbf{A}]_{n,m}$. The matrix \mathbf{I}_N denotes the identity matrix of dimension $N \times N$. The conjugate transpose of a matrix \mathbf{A} is expressed as \mathbf{A}^H . The cardinality of a set \mathcal{S} is given by $|\mathcal{S}|$. The notation $\text{diag}(\mathbf{A}_1, \dots, \mathbf{A}_N)$ denotes the block-diagonal matrix of matrices \mathbf{A}_i and $i = 1, \dots, N$. The relation symbol \succ denotes the matrix inequality on a cone of nonnegative definite matrices. The operator $\text{Tr}(\mathbf{A})$ denotes the trace operation on a square matrix \mathbf{A} . The space of non-negative real numbers is denoted by \mathbb{R}_+ .

II. SYSTEM SETUP AND DATA MODEL

The network has N cooperating BS arrays, each equipped with N_A antennas. At each time instant, N users of the index set \mathcal{U} , each equipped with a single antenna, are jointly served in this network. At a time instant one user per cell is scheduled; thus, there is one beamforming vector per cell or BS antenna-array. In unicast transmission, there is always one scheduled user per antenna array. A user i receives the signal:

$$r_i = \mathbf{h}_{i,i}^H \boldsymbol{\omega}_i s_i + \sum_{l \in \mathcal{U}, l \neq i} \mathbf{h}_{l,i}^H \boldsymbol{\omega}_l s_l + n_i. \quad (1)$$

Here, n_i is the interference plus noise of adjacent networks. The signal transmitted to user i is given by s_i with $\mathbb{E}\{|s_i|^2\} = 1$ and $\mathbb{E}\{s_l s_i^*\} = 0$ if $l \neq i$. The channel between the BS array of user l and user i is given by $\mathbf{h}_{l,i} \in \mathbb{C}^{N_A \times 1}$. The beamforming vector of the l th antenna array with N_A antenna elements is denoted by $\boldsymbol{\omega}_l = [\omega_l(0), \omega_l(1), \dots, \omega_l(N_A - 1)]^T$. With the assumption of Gaussian noise with $\mathbb{E}\{|n_i|^2\} = \sigma_i^2$ and $\mathbb{E}\{n_i\} = 0$, the instantaneous downlink SINR is

$$\gamma_i^D(\boldsymbol{\Omega}) = \frac{|\mathbf{h}_{i,i}^H \boldsymbol{\omega}_i|^2}{\sum_{\substack{l \in \mathcal{U} \\ l \neq i}} |\mathbf{h}_{l,i}^H \boldsymbol{\omega}_l|^2 + \sigma_i^2}. \quad (2)$$

In what follows, we define the downlink SINR distinguishing two different cases of CSI.

If **instantaneous CSI** is used, the following matrices normalized to the noise power, will be used:

$$\mathbf{R}_{l,i} = \frac{1}{\sigma_i^2} \mathbf{h}_{l,i} \mathbf{h}_{l,i}^H \quad i, l \in \mathcal{U}. \quad (3)$$

TABLE I: Overview of related work since 1998.

Year	Author	Problem	Power constr.	CSI	Theory, Solution
1998	Rashid-Farrokhi et al, [5]	PMP	-	instantaneous	Perron Frobenius theorem
1998	Yang, Xu, [6]	MBP	sum power	instantaneous	Perron Frobenius theorem
2001	Bengtsson, Ottersten, [1]	PMP	-	long-term	semidefinite programming
2002	Boche, Schubert, [10]	MBP	sum power	long-term	Perron Frobenius theorem
2005	Song, Cruz, Rao, [11]	PMP	-	instantaneous	Linear programming duality
2006	Wiesel, Eldar, Shamai, [8]	MBP	sum power	instantaneous	second order cone programming
2007	Yu, Lan, [12]	PMP	per-antenna	instantaneous	Lagrangian
2008	Tölli, Codreanu, Juntti, [3]	MBP	per-antenna	instantaneous	second order cone programming
2009	Tan, Chiang, Srikant [13], [14]	MBP	sum power	instantaneous	non-linear Perron Frobenius theo.
2009	Tan, Chiang, Srikant [15]	MBP	per array	instantaneous	Non-linear Perron Frobenius theo.
2011	Cai, Quek, Tan, Low, [16], [9], [17]	MBP	per array	instantaneous	Non-linear Perron Frobenius theo.
2011	Negro, Cardone, Ghauri, Slock, [18]	MBP	per array	instantaneous	Bisection over convex PMPs

TABLE II: Overview of related problems.

Problem	Objective	Power Constraints	CSI	Convexity / Strong Duality
PMP	sum power	-	long-term or instantaneous CSI	strong duality
PMP	sum power	per-antenna power	long-term or instantaneous. CSI	convex form
MBP	min. SINR	sum power	long-term or instantaneous CSI	strong duality
MBP	min. SINR	per-antenna	instantaneous CSI	quasi-convex form

Long-term CSI is often used in a multi-cell optimization due to the reduced CSI feedback rate. The assumption of this long-term CSI implies that only the mean SINR is considered. Here, an additional expectation over the channel realizations \mathcal{H} is taken. The result is the spatial correlation matrix given by:

$$\mathbf{R}_{l,i} = \frac{1}{\sigma_i^2} \mathbb{E}_{\mathcal{H}} \{ \mathbf{h}_{l,i} \mathbf{h}_{l,i}^H \} \quad i, l \in \mathcal{U}. \quad (4)$$

Downlink SINR: Using the definitions of the spatial correlation matrices (3) or (4), the SINR is then defined by:

$$\gamma_i^D(\boldsymbol{\Omega}) = \frac{\boldsymbol{\omega}_i^H \mathbf{R}_{i,i} \boldsymbol{\omega}_i}{\sum_{l \in \mathcal{U}, l \neq i} \boldsymbol{\omega}_l^H \mathbf{R}_{l,i} \boldsymbol{\omega}_l + 1}. \quad (5)$$

Note the spatial correlation matrices (4) are normalized by the noise power.

Uplink SINR: In addition to the downlink (DL) SINR (5), the uplink (UL) SINR can be defined as follows: let the UL receive beamforming vectors of a BS array be given by $\mathbf{v}_i \in \mathbb{C}^{N_A \times 1}$, with $\|\mathbf{v}_i\| = 1$. Then with the UL power $\lambda_i \geq 0$, the dual UL SINR of user i is defined by:

$$\gamma_i^U(\boldsymbol{\mu}, \boldsymbol{\lambda}, \mathbf{v}_i) = \frac{\lambda_i \mathbf{v}_i^H \mathbf{R}_{i,i} \mathbf{v}_i}{\mathbf{v}_i^H (\mathbf{M}_i + \sum_{l \in \mathcal{U}, l \neq i} \lambda_l \mathbf{R}_{i,l}) \mathbf{v}_i}. \quad (6)$$

The diagonal matrix $\mathbf{M}_i \succeq 0$, called UL-scaling matrix in this paper, depends on three power constraints and is defined below. The definition of the UL SINR can be used to solve the surrogate dual problem of the MBP in Section III-C.

Power constraints: Let $\boldsymbol{\Omega} = [\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_M]$ be the matrix containing all beamforming vectors, this paper considers three different power constraints:

- In the case of a **sum power constraint**, the total power of all transmitting stations is limited by P . The convex cone of beamforming vectors satisfying the sum power constraint is given by:

$$\mathcal{F} = \{ \boldsymbol{\Omega} \in \mathbb{C}^{N_A \times M} : \sum_{l \in \mathcal{U}} \boldsymbol{\omega}_l^H \boldsymbol{\omega}_l \leq P \}. \quad (7)$$

- If **per-BS antenna array power constraints** are used, each antenna array l of a BS will be subject to a total power budget P_l . This constraint is stricter and practically more relevant. The convex cone of beamforming vectors satisfying the per-BS antenna array power constraints is given by:

$$\mathcal{F} = \{ \boldsymbol{\Omega} \in \mathbb{C}^{N_A \times M} : \boldsymbol{\omega}_l^H \boldsymbol{\omega}_l \leq P_l \quad \forall l \in \mathcal{U} \}. \quad (8)$$

In the definition of the UL SINR (6), the UL-scaling matrix $\mathbf{M}_i = \mu_i \mathbf{I}_{N_A}$ for some $\mu_i \geq 0$ is used.

- If **Per-antenna power constraints** are given, the power of each antenna element with index a of the total set of \mathcal{A} is limited. It is assumed each BS array l has the same number N_A of antenna elements given by the set \mathcal{A}_l . The power of each antenna element a of BS array l is constrained by $P_{l,a}$.

The convex cone of beamforming vectors satisfying the per-antenna power constraints is given by:

$$\mathcal{F} = \{ \boldsymbol{\Omega} \in \mathbb{C}^{N_A \times 1M} : \|\boldsymbol{\omega}_l\|_a^2 \leq P_{l,a} \quad \forall a \in \mathcal{A}_l \quad \forall l \in \mathcal{U} \}. \quad (9)$$

In the definition of the UL SINR (6), the UL scaling matrix $\mathbf{M}_i = \text{diag}(\mu_{1,i}, \dots, \mu_{i,N_A})$ for all $\mu_{i,a} \geq 0$ is used.

III. OPTIMIZATION PROBLEM AND THE UPLINK-DOWNLINK DUALITY

We desire to improve the worst SINR of the currently scheduled users. The MBP can be stated as

$$\gamma^D = \max_{\boldsymbol{\Omega} \in \mathcal{F}} \min_{i \in \mathcal{U}} \gamma_i^D(\boldsymbol{\Omega}). \quad (10)$$

The beamforming matrix is given by $\boldsymbol{\Omega} = [\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_M]$. A balanced SINR can be the result. However, such an approach requires a smart selection of the set \mathcal{U} of active users, otherwise a single bad user can degrade the performance of all jointly active users. Different scheduling techniques

discussed in [23] can avoid such a situation. The problem can be simplified by using an additional slack variable γ :

$$\begin{aligned} \gamma^D &= \max_{\Omega \in \mathcal{F}, \gamma \in \mathbb{R}_+} \gamma \\ \text{s.t. } & \gamma_i^D(\Omega) \geq \gamma \quad \forall i \in \mathcal{U}. \end{aligned} \quad (11)$$

A. Semidefinite Relaxation

The MBP is non-convex for arbitrary spatial correlation matrices $\mathbf{R}_{l,i}$ and using general power constraints as per-BS antenna array power constraints (8). However, for all of these non-convex cases, the MBP can be relaxed to a quasi-convex program. The resulting convex feasibility check problem is a semidefinite program (SDP) [2]. A bisection over γ can iterate arbitrarily tightly to the optimal value if the bisection interval is correctly chosen. This approach is a standard method of solving the MBP and is used as a reference in this paper.

With $\mathbf{X} = [\mathbf{X}_1, \dots, \mathbf{X}_N]$ consisting of semidefinite matrices $\mathbf{X}_l = \boldsymbol{\omega}_l \boldsymbol{\omega}_l^H$, dropping the non-convex rank-1 constraint $\text{rank}(\mathbf{X}_l) = 1 \quad \forall l \in \mathcal{U}$, and fixing γ , the SDP is given by [24]:

$$\begin{aligned} \text{find } & \mathbf{X}, \\ \text{s.t. } & -\frac{1}{\gamma} \text{Tr}\{\mathbf{X}_i \mathbf{R}_{i,i}\} + \sum_{l \in \mathcal{U}, l \neq i} \text{Tr}\{\mathbf{X}_l \mathbf{R}_{l,i}\} + 1 \leq 0, \\ & \mathbf{X}_i \succeq 0, \quad \text{Tr}\{\mathbf{X}_i\} \leq P_i, \quad \forall i \in \mathcal{U}. \end{aligned} \quad (12)$$

$$(13)$$

In the case of a sum power constraint $\sum_{l \in \mathcal{U}} \text{Tr}\{\mathbf{X}_l\} = P$ instead of (13), the resulting matrices are proved to all have rank 1 and are, therefore, optimal [1]. With a per-BS antenna array or per-antenna element power constraints and arbitrary spatial correlation matrices, the solution has not been proved to be globally optimal in the literature.

B. Equivalent Quasi-Convex Form in the Case of Instantaneous CSI

In general, the problem (10) is non-convex because of the non-convex objective function over Ω :

$$f(\Omega) = \min_{i \in \mathcal{U}} \frac{\boldsymbol{\omega}_i^H \mathbf{R}_{i,i} \boldsymbol{\omega}_i}{\sum_{l \in \mathcal{U}, l \neq i} \boldsymbol{\omega}_l^H \mathbf{R}_{l,i} \boldsymbol{\omega}_l + 1}. \quad (14)$$

Note $f(\Omega)$ is a continuous function because the point-wise minimum of continuous functions is also continuous. However, if instantaneous CSI is available at the BSs, the MBP has an equivalent quasi-convex form for a sum power constraint, for per-antenna array and for per-antenna power constraints (see Proposition 1). It is desired to maximize (14); hence, the objective function must have an equivalent quasi-concave form to prove that the MBP has an equivalent quasi-convex form.

Definition 1: [2] A function $f(\mathbf{x})$ defined on a convex set \mathcal{F} is quasi-concave if every upper level set

$$\mathcal{S}_\alpha = \{\mathbf{x} \in \mathcal{F} : f(\mathbf{x}) \geq \alpha\} \quad (15)$$

of $f(\mathbf{x})$ is convex for every value of α .

Proposition 1: Let γ^D denote the solution of the MBP (10). Then the MBP (10) has an equivalent quasi-convex form with an optimal solution γ^{*D} for the given power constraints (7)-(9) and in the case of instantaneous CSI and matrices $\mathbf{R}_{l,i}$ defined in (3). Consequently $\gamma^D = \gamma^{*D}$ holds.

Proof: The point-wise minimum of a quasi-concave function is quasi-concave [2]. Therefore, only the upper level sets

$$\mathcal{S}_{\gamma,i} = \{\Omega \in \mathcal{F} : \frac{\boldsymbol{\omega}_i^H \mathbf{R}_{i,i} \boldsymbol{\omega}_i}{\sum_{l \in \mathcal{U}, l \neq i} \boldsymbol{\omega}_l^H \mathbf{R}_{l,i} \boldsymbol{\omega}_l + 1} \geq \gamma\} \quad (16)$$

must be convex. The same idea as, e.g., in [1], [12] is used to prove the convexity of the SINR constraint

$$\frac{1}{\gamma} \boldsymbol{\omega}_i^H \mathbf{R}_{i,i} \boldsymbol{\omega}_i \geq \sum_{l \in \mathcal{U}, l \neq i} \boldsymbol{\omega}_l^H \mathbf{R}_{l,i} \boldsymbol{\omega}_l + 1. \quad (17)$$

An arbitrary phase rotation of the beamforming vectors does not affect the SINR [1], if instantaneous CSI is given. Hence, the constraint (17) can be rewritten as in [1]:

$$\frac{1}{\sqrt{\gamma} \sigma_i} \mathbf{h}_{i,i}^H \boldsymbol{\omega}_i \geq \sqrt{\sum_{l \in \mathcal{U}, l \neq i} \boldsymbol{\omega}_l^H \mathbf{R}_{l,i} \boldsymbol{\omega}_l + 1}, \quad (18)$$

$$\text{with } \text{Im}\{\mathbf{h}_{i,i}^H \boldsymbol{\omega}_i\} = 0, \quad \mathbf{h}_{i,i}^H \boldsymbol{\omega}_i \geq 0, \quad (19)$$

if $\mathbf{R}_{l,i}$ is given by (3). The constraint (18) is a second order cone constraint [2] for a fixed parameter γ . With the SINR constraint (18) and per-antenna power constraints, the MBP (10) can be rewritten as a convex SOCP. ■

For a fixed (constant) γ , the feasibility check problem of the MBP can be expressed as a SOCP. The upper level sets of the equivalent form of the objective function are convex. Consequently, the objective function has an equivalent quasi-concave form. With the convex form of the SINR constraint (18) the MBP can be solved with the following convex feasibility check problem:

$$\begin{aligned} \text{find } & \Omega \\ \text{s.t. } & \frac{1}{\sqrt{\gamma}} \mathbf{h}_{i,i}^H \boldsymbol{\omega}_i \geq \sqrt{\sum_{l \in \mathcal{U}, l \neq i} \boldsymbol{\omega}_l^H \mathbf{R}_{l,i} \boldsymbol{\omega}_l + \sigma_i^2} \\ & |[\mathbf{w}_i]_a|^2 \leq P_{i,a} \quad \forall i, a. \end{aligned} \quad (20)$$

A bisection over γ can iterate arbitrarily closely to the global optimal value if the bisection interval is correctly chosen. In the case of per-antenna array power constraints, the feasibility check problem can be analogously derived. Note the convexity of the SINR constraints (17) for a given parameter γ is only proved for instantaneous CSI or spatial correlation matrices with rank 1. If arbitrary long-term CSI is used, the spatial correlation matrices (4) will have a higher rank. Therefore, this technique can not be used straightforwardly to prove the transformation to an equivalent quasi-convex problem.

C. Surrogate Duality for the MBP

This section introduces a new framework for a dual UL problem that is equivalent to the original MBP (10) if both, instantaneous CSI, or arbitrary long-term CSI in the form of higher rank spatial correlation matrices is available.

Section III-B illustrates that the MBP has equivalent quasi-convex form if instantaneous CSI is available. A quasi-convex problem can be solved directly by using the convex feasibility check problem of the primal problem. A simple bisection over

multiple convex feasibility check problems can be calculated to iterate as long as a precision¹ of ϵ is reached [2].

In [18], the authors propose a similar way to solve the unicast MBP. However, instead of the feasibility check problem of the primal problem, the dual UL problem of the PMP is used which is equivalent to the MBP for a given SINR γ . In [20], a direct solution based on the dual UL problem was proposed. The proposed iterative algorithm is derived based on the Lagrangian dual problem, which provides an upper bound of optimal solution. However, the MBP with more general power constraints, e.g., per-BS antenna array power constraints has merely an equivalent quasi-convex form. Therefore, the strong duality of the Lagrangian dual problem can not be proved so easily in this case. This section proposes an alternative and simpler framework for UL–DL duality based on the work on surrogate duality in quasi-convex programming of D. Luenberger [22].

Several publications on quasi-convex programming exist (e.g., [22]), and several dual problems have been proposed. Let $\mathcal{E} \subset \mathbb{Z}^n$ where \mathbb{Z}^n is an Euclidian space, in the case of a maximization problem [25], [26]

$$\begin{aligned} t_0 &= \max_{\mathbf{x} \in \mathcal{E}} f(\mathbf{x}) \\ \text{s.t. } & g_i(\mathbf{x}) \leq 0 \quad i = 1, \dots, N \end{aligned} \quad (21)$$

D. Luenberger [22] proposed a solution based on a so-called surrogate dual function

$$\begin{aligned} s(\boldsymbol{\mu}) &= \max_{\mathbf{x} \in \mathcal{E}} f(\mathbf{x}) \\ \text{s.t. } & \sum_{i=1}^N \mu_i g_i(\mathbf{x}) \leq 0, \quad \mu_i \geq 0 \quad i = 1, \dots, N. \end{aligned} \quad (22)$$

The vector $\boldsymbol{\mu}$ denotes the vector of the surrogate variables $\mu_i \geq 0$. With the definition of $\mathbf{g}(\mathbf{x}) = [g_1(\mathbf{x}), \dots, g_N(\mathbf{x})]^T$, he proves the following theorem:

Theorem 1: [22] Let $f(\mathbf{x})$ be a quasi-concave lower semi-continuous² objective function and let all $g_i(\mathbf{x})$ be convex, assume that $t_0 = \sup_{\mathbf{x} \in \mathcal{E}} \{f(\mathbf{x}) : \mathbf{g}(\mathbf{x}) \leq \mathbf{0}\}$ is finite. Then, $t_0 = \min_{\boldsymbol{\mu}} \{s(\boldsymbol{\mu})\}$, where the minimum is achieved for some $\boldsymbol{\mu} \geq \mathbf{0}$.

Note, instead of Luenberger's [22] minimization problem, this paper regards a maximization problem, hence, the reversals of the minimization and maximization can be used [25]. Here, the surrogate dual aims at finding the tightest upper bound [25] instead of the tightest lower bound as in [22]. Theorem 1 requires a quasi-convex objective function. The following theorem generalizes the work of Luenberger:

Theorem 2: [25], [27] If an \mathbf{x}^* solves (22) for a $\boldsymbol{\mu}^* \in \mathbb{R}_+^n$ and \mathbf{x}^* is feasible in (21), then \mathbf{x}^* solves (21) and $t_0 = \min_{\boldsymbol{\mu}} \{s(\boldsymbol{\mu})\}$.

A fundamental result in the surrogate duality theory is proved in [25]:

Theorem 3: [25] The surrogate dual function $s(\boldsymbol{\mu})$ is quasi-convex in $\boldsymbol{\mu}$.

¹In this paper, the term precision means the residual distance to the optimal solution.

²Along lines, e.g., every concave function is lower semi-continuous along lines [22].

Hence, a global minimizer over $\boldsymbol{\mu}$ can always attain the minimum. Furthermore, the surrogate dual problem is connected with the Lagrangian dual problem. Greenberg et. al. have proved the following theorem [25]:

Theorem 4: [25], [27] The solution of the surrogate dual problem $\min_{\boldsymbol{\mu}} s(\boldsymbol{\mu})$ is a tighter bound than the solution of the Lagrangian dual $\min_{\boldsymbol{\mu}} l(\boldsymbol{\mu})$; hence, $\min_{\boldsymbol{\mu}} s(\boldsymbol{\mu}) \leq \min_{\boldsymbol{\mu}} l(\boldsymbol{\mu})$. If $\min_{\boldsymbol{\mu}} s(\boldsymbol{\mu}) = \min_{\boldsymbol{\mu}} l(\boldsymbol{\mu})$ then there exists an \mathbf{x} such that $\boldsymbol{\mu}^T \mathbf{g}(\mathbf{x}) = 0$. Hence, the surrogate constraint is satisfied with equality.

This duality theory is used in this section to derive a dual UL problem of the unicast MBP. The multiple power constraints are combined to a single power constraint as shown in (22). At this point it could be helpful to give a short overview of the propositions for the duality proof in this paper:

- The combination of multiple power constraints to a single weighted sum power constraint results in a UL MBP with an inner and an outer problem. The inner problem corresponds to a DL MBP with a weighted sum power constraint. Transformation of this problem to the UL domain leads to a quasi-convex problem. The Lagrangian dual UL problem of this problem is presented in Lemma 1. Strong duality is proved for the sum power constrained case, consequently, the Lagrangian duality is tight.
- In Proposition 2, the surrogate dual function is derived by using the previous result in Lemma 1. The surrogate dual function combines multiple power constraints to a single weighted sum power constraint. This surrogate function is transformed to the UL domain where it can be solved efficiently. The surrogate function in the UL domain leads to the surrogate dual problem of the original MBP.
- The question is now whether this surrogate dual problem solves the MBP with multiple power constraints. In the case of instantaneous CSI or rank-1 spatial correlation matrices, an equivalent quasi-convex form exists. Based on the equivalent quasi-convex form, strong (surrogate) duality can be directly proved with the duality theorem of Luenberger (Theorem 1). This is proved in Proposition 4.
- If arbitrary long-term CSI in the form of higher-rank spatial correlation matrices is given no equivalent quasi-convex form is known. Thus, Theorem 1 can not be applied. Proposition 5 represents an alternative proof for strong duality in this more general case based on Theorem 2.
- With the help of Theorem 3, Proposition 6 states that the surrogate dual problem is quasi-convex; hence, a global optimal value exists.
- Proposition 7 shows the equivalence of the surrogate dual and the Lagrangian dual of the original MBP with general power constraints. According to Theorem 4, it follows that the weighted sum power constraint is satisfied with equality.

The derivation of the surrogate dual problem is based on an inner MBP with a weighted sum power constraint and its Lagrangian dual problem, which is tight if a sum power constraint is given. Consider the following unicast DL MBP

where the weighted sum power is limited to P :

$$f^D(\boldsymbol{\mu}) = \max_{\Omega} \min_{i \in \mathcal{U}} \gamma_i^D(\Omega) \quad (23)$$

$$\text{s.t. } \sum_{i \in \mathcal{U}} \boldsymbol{\omega}_i^H \mathbf{M}_i \boldsymbol{\omega}_i \leq P. \quad (24)$$

where $\mathbf{M}_i = \text{diag}(\mu_{1,i}, \dots, \mu_{i,N_A}) \succeq 0$ in the case of per-antenna array element power constraints and $\mathbf{M}_i = \mu_i \mathbf{I}_{N_A} \succeq 0$ in the case of per-BS antenna array power constraints. The vector $\boldsymbol{\mu}$ is the vector of all $\mu_{a,i}$ s or μ_i s. For a fixed $\boldsymbol{\mu}$, this problem is a MBP with a weighted sum power constraint. The Lagrangian dual UL problem is given by the following lemma:

Lemma 1: With the definition of the UL SINR (6), the Lagrangian dual of the unicast DL MBP (23), (24) is given by:

$$f^U(\boldsymbol{\mu}) = \max_{\boldsymbol{\lambda}, \mathbf{V}} \min_{i \in \mathcal{U}} \gamma_i^U(\boldsymbol{\mu}, \boldsymbol{\lambda}, \mathbf{v}_i) \quad (25)$$

$$\text{s.t. } \sum_{i \in \mathcal{U}} \lambda_i \leq P \quad \lambda_i \geq 0, \quad \forall i \in \mathcal{U}. \quad (26)$$

At the optimum, the power constraints of problems (23) and (25) are satisfied with equality and both problems have the same SINR $f^U(\boldsymbol{\mu}) = f^D(\boldsymbol{\mu})$.

Proof: The derivation of the Lagrangian dual is a simple extension of [28]. The proof of the strong duality is an extension of [4, Lemma 2]. [9] presents the duality in the case of instantaneous CSI or rank-1 spatial correlation matrices and per-BS power constraints. The proof for the case of per-antenna power constraints is a straightforward extension and presented in Appendix A. ■

Proposition 2: With the diagonal matrices $\mathbf{P}_i = \text{diag}(P_{i,1}, \dots, P_{i,N_A})$ in the case of per-antenna power constraints, or diagonal matrices $\mathbf{P}_i = \frac{P_i}{N_A} \mathbf{I}_{N_A}$ in the case of per-BS array power constraints, the surrogate dual function (or surrogate problem) of the unicast DL MBP (10) with general (per-antenna or per-antenna array) power constraints is given by:

$$s^U(\boldsymbol{\mu}) = \max_{\boldsymbol{\lambda}, \mathbf{V}} \min_{i \in \mathcal{U}} \gamma_i^U(\boldsymbol{\mu}, \boldsymbol{\lambda}, \mathbf{v}_i) \quad (27)$$

$$\text{s.t. } \sum_{i \in \mathcal{U}} \lambda_i \leq \sum_{i \in \mathcal{U}} \text{Tr}\{\mathbf{M}_i \mathbf{P}_i\} \\ \lambda_i \geq 0, \quad \mathbf{M}_i \succeq 0, \quad \forall i \in \mathcal{U}$$

for a $\boldsymbol{\mu} \geq 0$ and $\boldsymbol{\mu} \neq \mathbf{0}$.

Proof: The surrogate dual function of the MBP with per-antenna power constraints is given by:

$$s^D(\boldsymbol{\mu}) = \max_{\Omega} \min_{i \in \mathcal{U}} \gamma_i^D(\Omega) \quad (28)$$

$$\text{s.t. } p_{\Sigma}(\boldsymbol{\mu}, \Omega) \leq 0. \quad (29) \\ \mathbf{M}_i \succeq 0 \quad \forall i \in \mathcal{U}.$$

With the diagonal matrices $\mathbf{P}_i = \text{diag}(P_{i,1}, \dots, P_{i,N_A})$, the per-antenna power constraints are combined to the weighted sum power constraint:

$$p_{\Sigma}(\boldsymbol{\mu}, \Omega) = \sum_{i \in \mathcal{U}} \sum_{a \in \mathcal{A}_i} \mu_{i,a} (|\boldsymbol{\omega}_i|_a|^2 - P_{i,a}) \leq 0 \\ \Leftrightarrow \sum_{i \in \mathcal{U}} \boldsymbol{\omega}_i^H \mathbf{M}_i \boldsymbol{\omega}_i \leq \sum_{i \in \mathcal{U}} \text{Tr}\{\mathbf{M}_i \mathbf{P}_i\}.$$

With the diagonal matrices $\mathbf{P}_i = \frac{P_i}{N_A} \mathbf{I}_{N_A}$, the per-antenna array power constraints are combined to the weighted sum power constraint:

$$p_{\Sigma}(\boldsymbol{\mu}, \Omega) = \sum_{i \in \mathcal{U}} \mu_i (\boldsymbol{\omega}_i^H \boldsymbol{\omega}_i - P_i) \leq 0 \\ \Leftrightarrow \sum_{i \in \mathcal{U}} \boldsymbol{\omega}_i^H \mathbf{M}_i \boldsymbol{\omega}_i \leq \sum_{i \in \mathcal{U}} \text{Tr}\{\mathbf{M}_i \mathbf{P}_i\}.$$

Thus, the problem (28), (29) can be stated as:

$$s^D(\boldsymbol{\mu}) = \max_{\Omega} \min_{i \in \mathcal{U}} \gamma_i^D(\Omega) \quad (30)$$

$$\text{s.t. } \sum_{i \in \mathcal{U}} \boldsymbol{\omega}_i^H \mathbf{M}_i \boldsymbol{\omega}_i \leq \sum_{i \in \mathcal{U}} \text{Tr}\{\mathbf{M}_i \mathbf{P}_i\} \quad (31) \\ \mathbf{M}_i \succeq 0, \quad \forall i \in \mathcal{U}.$$

With $P = \sum_{i \in \mathcal{U}} \text{Tr}\{\mathbf{M}_i \mathbf{P}_i\}$ and Lemma 1, the problem (30) can be transformed to the UL domain and the problem (27) is the result. ■

Proposition 3: If instantaneous CSI in the form of (3) is given, let $s^{*D}(\boldsymbol{\mu})$ denote the surrogate dual of the equivalent quasi-convex form of the original MBP (10), then $s^{*D}(\boldsymbol{\mu}) = s^D(\boldsymbol{\mu})$ holds.

Proof: The proof is a straightforward extension of the theory in [8]. ■

Assuming there exists an equivalent quasi-convex form of the MBP (10), the proof of strong duality is straightforward with the help of Theorem 1.

Proposition 4: If the primal unicast DL MBP (10) has an equivalent quasi-convex form which has the solution γ^{*D} , the optimal solution γ^D of (10) is given by:

$$\gamma^D = \gamma_S^U = \min_{\boldsymbol{\mu}} s^U(\boldsymbol{\mu}) \quad (32)$$

for a $\boldsymbol{\mu} \neq \mathbf{0}$ and $\boldsymbol{\mu} \geq 0$.

Proof: Problem (35) is the surrogate dual problem. Proposition 2 declares the lack of a duality gap for the inner problem hence,

$$s^D(\boldsymbol{\mu}) = s^U(\boldsymbol{\mu})$$

holds. Consequently, also

$$\gamma_S^D = \min_{\boldsymbol{\mu}} s^D(\boldsymbol{\mu}) = \min_{\boldsymbol{\mu}} s^U(\boldsymbol{\mu}) = \gamma_S^U \quad (33)$$

holds. According to the duality theorem of Luenberger (Theorem 1), strong duality between the surrogate dual problem and the primal problem holds if the primal problem is quasi-convex, assuming γ^{*D} is the solution of the equivalent quasi-convex form of the MBP (10) and $s^{*D}(\boldsymbol{\mu})$ is the equivalent quasi-convex form of the surrogate function. Due to (33), also

$$\gamma^{*D} = \min_{\boldsymbol{\mu}} s^{*D}(\boldsymbol{\mu}) = \min_{\boldsymbol{\mu}} s^U(\boldsymbol{\mu}) = \gamma_S^{*U} \quad (34)$$

holds. Since $s^{*D}(\boldsymbol{\mu}) = s^D(\boldsymbol{\mu})$, $s^{*U}(\boldsymbol{\mu}) = s^U(\boldsymbol{\mu})$, $\gamma_S^{*U} = \gamma_S^U$, and $\gamma^{*D} = \gamma^D$ holds,³ (32) holds as well. ■

³In the case of instantaneous CSI and a weighted sum power constraint or per-antenna or per-antenna array power constraints, the MBP has an equivalent quasi-convex form (see Propositions 1 and 3).

Given arbitrary higher rank spatial correlation matrices as (4), the proof of strong duality is not straightforward, because, then, no equivalent quasi-convex form of the MBP is known. However, the following proposition formulates the general duality result for the case of higher rank spatial correlation matrices and per-BS array power constraints:

Proposition 5: Given arbitrary higher rank spatial correlation matrices as (4), the solution of the dual UL problem is equal to the solution of the primal unicast DL MBP (10). Hence, with $\boldsymbol{\mu} \neq \mathbf{0}$ and $\gamma_S^U = \gamma^D$, the solution is given by

$$\gamma_S^U = \min_{\boldsymbol{\mu}} s^U(\boldsymbol{\mu}). \quad (35)$$

Proof: Due to $s^D(\boldsymbol{\mu}) = s^U(\boldsymbol{\mu})$, also

$$\gamma_S^D = \min_{\boldsymbol{\mu}} s^D(\boldsymbol{\mu}) = \min_{\boldsymbol{\mu}} s^U(\boldsymbol{\mu}) = \gamma_S^U \quad (36)$$

holds. The question is whether $\gamma_S^D = \gamma^D$. If an $\boldsymbol{\Omega}^*$ is feasible in $s^D(\boldsymbol{\mu}^*)$, there must be a $\boldsymbol{\mu}^* \geq \mathbf{0}$ and $\boldsymbol{\mu}^* \neq \mathbf{0}$ which is also feasible in (10) according to Theorem 2. We must show the feasibility of $\boldsymbol{\Omega}^*$ in the original problem so that $\boldsymbol{\omega}_k^{*H} \boldsymbol{\omega}_k^* \leq P_k \forall k \in \mathcal{U}$. Let $\boldsymbol{\mu}_k^* \in \mathbb{R}^{M \times 1}$ be a vector with exactly one non-zero element, $\mu_k^* > 0$. Then, the surrogate problem is

$$s^D(\boldsymbol{\mu}_k^*) = \max_{\boldsymbol{\Omega}} \min_{i \in \mathcal{U}} \gamma_i^D(\boldsymbol{\Omega}) \quad (37)$$

$$\text{s.t. } \boldsymbol{\mu}_k^{*H} \boldsymbol{\omega}_k^H \boldsymbol{\omega}_k \leq \mu_k^* P_k \quad \mu_k > 0. \quad (38)$$

where (38) is finally $\boldsymbol{\omega}_k^H \boldsymbol{\omega}_k \leq P_k$. The k th constraint of the original problem (10) is satisfied if the constraint (38) of surrogate problem $s^D(\boldsymbol{\mu}_k^*)$ is satisfied. As in [17, Theorem 4], let \mathcal{F} denotes the feasible set of the original problem (10), and

$$\mathcal{F}_k^D = \{\boldsymbol{\Omega}_k^* \in \mathbb{C}^{N_A \times M} : \boldsymbol{\omega}_k^H \boldsymbol{\omega}_k \leq P_k\}$$

denotes the feasible set of (37), (38), then $\mathcal{F} = \bigcap_{k \in \mathcal{U}} \mathcal{F}_k^D$. The index $k^* = \operatorname{argmin}_{k \in \mathcal{U}} \{s^D(\boldsymbol{\mu}_k^*)\}$ is the index where all power constraints are satisfied. According to Theorem 2, $\boldsymbol{\Omega}_{k^*}^* \in \mathcal{F}$ solves the surrogate dual problem for a $\boldsymbol{\mu}_{k^*}^* \geq \mathbf{0}$, $\boldsymbol{\mu}_{k^*}^* \neq \mathbf{0}$ and is also feasible in the primal problem (10). Consequently, according to Theorem 2, the solution of the surrogate dual problem is a tight upper bound. ■

The question is now whether (35) attains the optimal value. In [25], the authors prove that the surrogate dual provides a tighter bound than the Lagrangian dual. The following proposition declares the quasi-convexity of the surrogate dual problem.

Proposition 6: The surrogate dual problem (35) can be globally optimally solved.

Proof: Due to Theorem 3, the surrogate dual function is quasi-convex, hence a global minimizer can attain the value of the global minimum. ■

In [29], the authors discuss the Pareto optimality of the achievable rate. This paper regards the point on where all SINRs are balanced. If a balanced SINR exists, the Lagrangian dual and the surrogate dual of the MBP (10) are equivalent.

Definition 2: A tuple of SINRs $(\gamma_1^D, \gamma_2^D, \dots, \gamma_M^D)$ is balanced if $\gamma_1^D = \gamma_2^D = \dots = \gamma_M^D$.

A balanced SINR may not exist if, e.g., a user does not receive any interference from other BSs in the network; hence the network is not interference coupled. More details concerning conditions for a balanced SINR are explained in [30], [31].

Proposition 7: If a balanced SINR according to Definition 2 exists, the Lagrangian dual of the MBP (10) is given by

$$\begin{aligned} \gamma_L^U &= \min_{\boldsymbol{\mu}} \max_{\boldsymbol{\lambda}, \mathbf{V}} \min_{i \in \mathcal{U}} \gamma_i^U(\boldsymbol{\mu}, \boldsymbol{\lambda}, \mathbf{v}_i) \\ &\text{s.t. } \sum_{i \in \mathcal{U}} \lambda_i \leq \sum_{i \in \mathcal{U}} \operatorname{Tr}\{\mathbf{M}_i \mathbf{P}_i\}, \\ &\lambda_i \geq 0, \quad \mathbf{M}_i \succcurlyeq \mathbf{0}, \quad \forall i \in \mathcal{U}. \end{aligned} \quad (39)$$

and at the optimum, the weighted sum power constraint (31) is satisfied with equality.

Proof: The derivation of the Lagrangian is presented in Appendix B.

The surrogate dual (35) is equivalent to the Lagrangian dual (39), $\gamma_L^U = \gamma_S^U$. According to Theorem 4, the weighted sum power constraint (31) is satisfied with equality. ■

IV. ITERATIVE ALGORITHM

The structure of the dual UL problem offers a solution based on simple mathematical operations. The solution consists of an outer minimization over $\boldsymbol{\mu}$ (35) and an inner maximization over $\boldsymbol{\lambda}$ and \mathbf{V} (27). Thus, two loops (an inner and an outer loop) as in [12] are used in what follows. In contrast to [12], the presented approach jointly finds the balanced SINR γ .

The duality of the inner UL problem (27) with the DL MBP is proved in Lemma 1. The problem (23), (24) corresponds to a MBP with a weighted sum power constraint. For the problems (23), (24) and (25), (26) strong duality, if a weighted sum power constraint is given, is proved (Lemma 1). The corresponding UL problem (25), (26) is reduced to a computation of the largest eigenvalue and the corresponding eigenvector. This paper presents an iterative computation of the largest eigenvector instead of an eigenvector decomposition. In [32], the so-called power iteration is shown to be a low complexity solution if only the largest eigenvalue is desired.

Regarding the surrogate dual problem (35), the inner problem in the downlink domain for fixed $\boldsymbol{\mu}$ is a MBP as in [9], [13], with a weighted sum power constraint, where each beamforming vector is scaled by \mathbf{M}_i . However, in contrast to [9] and [13], the CSI is given here in the form of higher rank spatial correlation matrices. In the following section (IV-A), a low complexity method based on an iterative computation of the UL beamforming vectors and the UL power $\boldsymbol{\lambda}$ for a fixed vector $\boldsymbol{\mu}$ is presented. The next section (IV-B) presents a solution, similar to [12], where the optimal vector $\boldsymbol{\mu}$ is found by a subgradient projection algorithm in an outer loop. However, it is known that a subgradient projection method requires a properly chosen step size; otherwise, the convergence is very slow [12]. Therefore, a faster converging low complexity method based on a simple scaling of the $\mu_{i,a}$ s or $\mu_{i,a}$ s is presented as well. The iterative algorithm has a lower complexity than the convex solver based methods, especially for a low precision.

A. Inner loop

The inner function corresponds to a MBP with a weighted sum power constraint. The inner maximization in the dual UL problem (25) of (23) is optimized over λ and \mathbf{V} . This is done iteratively in the inner loop by first computing the optimal receive beamforming vectors \mathbf{v}_i s and then updating the optimal UL power allocation λ for fixed \mathbf{V} until convergence is achieved. In [8] and [13], where instantaneous CSI is used, a fixed point iteration for the λ vector is introduced. The convergence is proved in [9] with the help of [33]. The authors of [9] use instantaneous CSI to balance the SINR. Using these resulting rank-1 matrices, an optimal closed form solution is given by the minimum variance distortionless response (MVDR) beamformers. For a given λ vector, a closed form solution for the UL beamformers exists. In contrast to that, the MBP is based here on higher rank spatial correlation matrices. Consequently, no closed form solution exists. A solution based on higher rank spatial correlation matrices for a sum power constraint MBP is already proposed in [4]. In [28], a method with reduced complexity compared to [4] is proposed. In [28], in each iteration a complete matrix inversion of the interference matrix

$$\Sigma_i = \mathbf{M}_i + \sum_{l \in \mathcal{U}, l \neq i} \lambda_l \mathbf{R}_{i,l} \quad (40)$$

and an eigenvalue decomposition is calculated. The inner problem (25) of the surrogate dual problem is the maximization of the balanced UL SINR (6) and is equivalent to the eigenproblem

$$\mathbf{R}_{i,i} \mathbf{v}_i = \chi_{i,n} \Sigma_i \mathbf{v}_i, \quad (41)$$

with the largest eigenvalue $\chi_{i,max} = \max_{1 \leq n \leq N_A} (\chi_{i,n}) = (\lambda_i)^{-1}$. The matrix Σ_i is regular, hence, the generalized eigensystem can be transformed to a special eigensystem [32] and the UL power can be directly computed as in [28]:

$$\lambda_i = \frac{1}{\chi_{i,max}(\Sigma_i^{-1} \mathbf{R}_{i,i})}. \quad (42)$$

The convergence of the resulting fixed point iteration is proved in [9].

This paper proposes a further complexity reduction, as a complete eigenvalue decomposition is not necessary. The eigenvalues corresponding to the UL power and the eigenvectors corresponding to the UL beamformers can be computed iteratively and jointly with the uplink power computation. Hence, compared to [28], less complexity per iteration is achieved, because instead of an eigenvalue decomposition, just a matrix vector multiplication $\mathbf{v}_i = \Sigma_i^{-1} \mathbf{R}_{i,i} \mathbf{v}_i$ is performed. If $(\lambda_i)^{-1}$ is strictly the largest eigenvalue, the eigensystem can be solved iteratively by the power iteration [32]. The inverse of the largest eigenvalue $\chi_{i,max}(\Sigma_i^{-1} \mathbf{R}_{i,i})$ corresponds to UL power λ_i which is computed for fixed \mathbf{V} directly by:

$$\lambda_i = \frac{\mathbf{v}_i^H \Sigma_i \mathbf{v}_i}{\mathbf{v}_i^H \mathbf{R}_{i,i} \mathbf{v}_i}. \quad (43)$$

The power iteration is a low complex algorithm for finding the dominant eigenvector of an eigensystem [32]. It is presented in Algorithm 1. As in [13] and [28], the λ_i s are scaled such

that the constraint $\sum_{i \in \mathcal{U}} \lambda_i \leq \sum_{i \in \mathcal{U}} \text{Tr}\{\mathbf{M}_i \mathbf{P}_i\}$ of the dual problem in (27) is satisfied with equality.

Algorithm 1 Inner loop: vector iteration

```

repeat
  for  $i = 1$  to  $M$  do
     $\mathbf{v}_i \leftarrow [\mathbf{M}_i + \sum_{l \in \mathcal{U}, l \neq i} \lambda_l \mathbf{R}_{i,l}]^{-1} \mathbf{R}_{i,i} \mathbf{v}_i$ 
    Set  $\|\mathbf{v}_i\| = 1 \quad \forall i \in \mathcal{U}$ 
     $\lambda_i \leftarrow \frac{\mathbf{v}_i^H [\mathbf{M}_i + \sum_{l \in \mathcal{U}, l \neq i} \lambda_l \mathbf{R}_{i,l}] \mathbf{v}_i}{\mathbf{v}_i^H \mathbf{R}_{i,i} \mathbf{v}_i}$ 
  end for
   $\lambda_i = \beta \tilde{\lambda}_i, \forall i \in \mathcal{U}$ , with  $\beta = \frac{\sum_{i \in \mathcal{U}} \text{Tr}\{\mathbf{M}_i \mathbf{P}_i\}}{\sum_{i \in \mathcal{U}} \lambda_i}$ 
until convergence
 $\mathbf{p} = (\frac{1}{\gamma} \mathbf{D}^{-1} - \Psi)^{-1} \mathbf{1}$ 
return  $\mathbf{V}, \gamma_{opt}^U = \beta$ 

```

If there exists a unique spectral radius $\rho(\Sigma_i^{-1} \mathbf{R}_{i,i}) = \max_{1 \leq n \leq N_A} (\chi_{i,n}(\Sigma_i^{-1} \mathbf{R}_{i,i}))$ of the matrix $\Sigma_i^{-1} \mathbf{R}_{i,i}$, the power iteration will converge. The convergence is geometric with a ratio of the largest eigenvalue to the second largest eigenvalue [32]. If the largest eigenvalue is significantly larger than the second largest eigenvalue, the convergence of the inner loop is very fast. The numerical results suggest that the case of multiple identical eigenvalues does not occur. The inner loop converges after 29 iterations for a precision of 10^{-5} .

After obtaining the normalized UL beamformer, the DL beamforming weights can be obtained by $\mathbf{w}_i = \sqrt{p_i} \mathbf{v}_i$. In Appendix A, the DL power vector \mathbf{p} is derived and given by

$$\mathbf{p} = (\frac{1}{\gamma} \mathbf{D}^{-1} - \Psi)^{-1} \mathbf{1} \quad (44)$$

if $\gamma = \gamma^D = \gamma^U$. Algorithm 1 has similarities to the solution proposed in [9]. However, the proposed solution is also able to solve the MBP if long-term CSI in the form of higher rank spatial correlation matrices is available.

B. Outer loop

The outer loop minimizes $f^D(\boldsymbol{\mu})$ (23) such that the given transmit power constraints are met. In this paper, two methods for updating the $\boldsymbol{\mu}$ vector or the \mathbf{M}_i matrices are proposed:

- 1) **Method 1: Subgradient projection method:** The update of the $\boldsymbol{\mu}$ vector is based on the subgradient method such that the power constraint $\sum_{i \in \mathcal{U}} \text{Tr}\{\mathbf{M}_i \mathbf{P}_i\} \leq \sum_{i \in \mathcal{U}} \boldsymbol{\omega}_i^H \mathbf{M}_i \boldsymbol{\omega}_i$ is satisfied [20], [16]. This method is similar to the subgradient projection method proposed in [12].
- 2) **Method 2: Low complexity μ -scaling (μ -SC):** In the constraint (31), the two sums are weighted sums over $\boldsymbol{\mu}$. Since, the constraint is satisfied with equality at the optimum (Proposition 7),

$$\sum_{i \in \mathcal{U}} \text{Tr}\{\mathbf{M}_i \mathbf{P}_i\} = \sum_{i \in \mathcal{U}} \boldsymbol{\omega}_i^H \mathbf{M}_i \boldsymbol{\omega}_i \quad (45)$$

the \mathbf{M}_i s can be updated by comparing the power coupled with each \mathbf{M}_i with \mathbf{P}_i and then scaling the $\mu_{i,a}$ s such that the constraint in (31) is satisfied with equality. Let

$\hat{\mathbf{M}}_i$ be the value of \mathbf{M}_i of the previous iteration, the \mathbf{M}_i s are computed by the following update in the case of per-antenna power constraints:

$$\begin{aligned} \tilde{\mathbf{M}}_i &= \text{diag}(\boldsymbol{\omega}_i \boldsymbol{\omega}_i^H) \mathbf{P}_i^{-1} \hat{\mathbf{M}}_i, \\ \mathbf{M}_i &= \frac{\mathbf{1}^T \mathbf{P}}{\sum_{i \in \mathcal{U}} \text{Tr}\{\tilde{\mathbf{M}}_i \mathbf{P}_i\}} \tilde{\mathbf{M}}_i. \end{aligned} \quad (46)$$

In the case of per-BS antenna array power constraints the update is similar:

$$\tilde{\mathbf{M}}_i = \frac{[\mathbf{P}]_i}{P_i} \hat{\mathbf{M}}_i, \quad \mathbf{M}_i = \frac{\mathbf{1}^T \mathbf{P}}{\sum_{i \in \mathcal{U}} \text{Tr}\{\tilde{\mathbf{M}}_i \mathbf{P}_i\}} \tilde{\mathbf{M}}_i. \quad (47)$$

The update of Method 2 is based on the decoupling as in (37) and (38), where each \mathbf{M}_i is optimized independently. For fixed $\boldsymbol{\Omega}$, the update (46), (47) is a normalized affine selfmapping satisfying the sum power constraint (45). [33, Theorem 1] proves that a normalized selfmapping converges if the mapping is concave or affine. With (45), the mappings (46), (47) are affine in $\boldsymbol{\mu}$, hence according to [33, Theorem 1], the iteration converges for fixed $\boldsymbol{\Omega}$.

The outer loop is shown in Algorithm 2. With the updates (46), (47), a convergence is given in the case a balanced DL SINR exists.

Algorithm 2 Outer loop: DL Power and iterations over $\boldsymbol{\mu}$

```

Initialize  $\boldsymbol{\mu} = \mathbf{1}$ 
repeat
  Inner loop (Algorithm 1)
  Update the  $\boldsymbol{\mu}$  vector by Method 1 or 2.
until Convergence
return  $\boldsymbol{\Omega}$ 
    
```

C. Complexity

The work [28] presents a complexity analysis of the inner loop with an eigenvalue decomposition. In this paper, the complexity of the inner loop is further reduced by replacing the eigenvalue decomposition with a power iteration method [32]. In Appendix C, the computation of the flop count is estimated. Assuming K_I is the number of iterations of the inner loop and K_O is the number of iterations of the outer loop, the upper bound of the complexity of the proposed iterative algorithm with update (46) and with N users and N_A antenna elements per BS is in the order of $\mathcal{O}(K_O K_I N^2 (N_A^3 + N_A^2) + K_O N^3 N_A^2)$. The complexity of the bisection with a SDP is in the order of $\mathcal{O}(K_O \log(1/\epsilon) \sqrt{N N_A} (N^3 N_A^6 + N^2 N_A^2))$.

V. NUMERICAL RESULTS

In Table III, the main simulation parameters for the network are summarized. The numerical results are based on long-term CSI in the form of higher rank spatial correlation matrices. The power angular density distribution is assumed to be Laplacian [35] and a similar simulation setup compared to [34] is used here. The users are randomly distributed in the multicell network.

TABLE III: Simulation parameters.

Number of user drops	1000
Number of users per user drop	21
Number of BSs drop	21
Transmit antenna arrays	uniform linear arrays
Number of antenna array elements at BS	4
Number of antenna array elements at MS	1
Intersite distance	2000 m
Antenna spacing	half wavelength
Path loss exponent	3.76
Available CSI	long-term CSI
Power angular density	Laplacian [34]
Power constraint	per-array

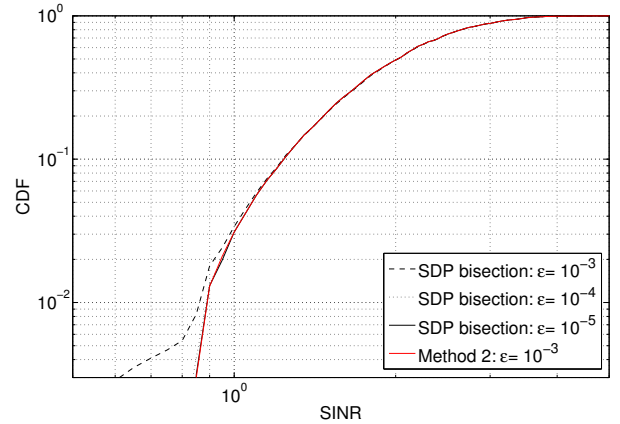


Fig. 1: Cumulative distribution function (CDF) of the SINR of the new iteration based method of Section IV (red) and the conventional SDP based bisection based method of Section III-A. The scaling of both axes is logarithmic.

Three algorithms are compared in this section:

- The bisection algorithm with a SDP as feasibility check problem given in Section III-A.
- The new iterative algorithm of Section IV with Method 1 (subgradient method).
- The new iterative algorithm of Section IV with Method 2 (46), (47).

Regarding Figure 1, the approach based on the interior point method to solve the SDP [36] requires a higher precision for the bisection algorithm to find an optimally balanced solution. As the precision increases, the solution for the SDP found by the interior point method improves (see Figure 2). The optimality ratio is the ratio of the largest eigenvalue of the solution matrices of the SDP divided by the sum of all eigenvalues of the solution matrix. The solution is nearly optimal at a precision value smaller than 10^{-5} .

Figure 1 shows that the iterative method is already very close to the optimum for a low precision around $\epsilon = 10^{-3}$. At a precision of $\epsilon = 10^{-5}$, the solution of the interior point method for the SDP is nearly optimal (see Figure 2). The new iterative algorithm (Section IV) to determine the \mathbf{M}_i matrices iterates over a MBP with a weighted sum power constraint. In each iteration, the matrices \mathbf{M}_i are determined so that, e.g., the per-antenna array power constraints are satisfied. Hence, the balanced SINR decreases per iteration. This convergence behavior can also be observed in Figure 3.

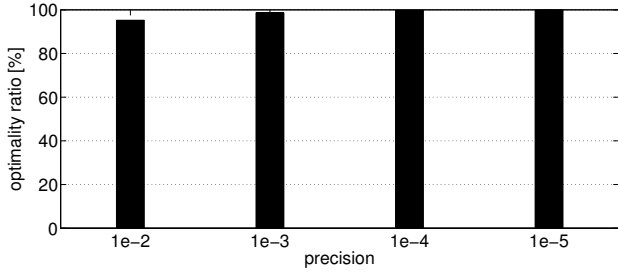


Fig. 2: Optimality ratio of the SDP based bisection method (Section III-A) for different precisions.

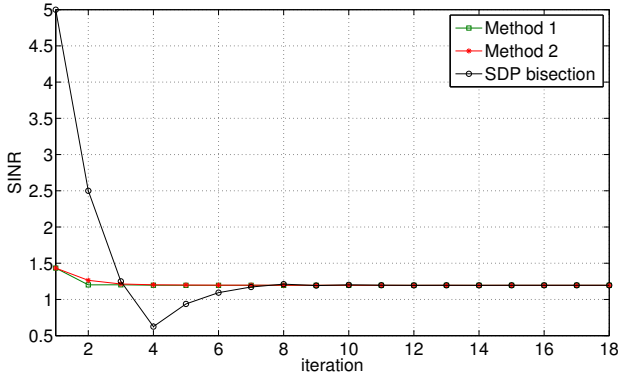


Fig. 3: Exemplified convergence behavior of the three different algorithms.

The interior point methods for the SDP perform not sufficiently well for a higher precision and, therefore, there are often users with a SINR below the optimum. An advantage of the presented solution is the convergence behavior. For a high precision, after a few iterations, the found SINR is already close to the lower bound or the optimal value. In Figure 3 the convergence behavior of the presented algorithm is depicted for an exemplified user drop. After a few iterations the algorithm is very close to the optimal value. The algorithm based on the subgradient based outer update can converge very fast if the step size is correctly chosen. This is not always the case.

The outer loop of the presented algorithm is solved by the subgradient projection method or by the μ -SC method (46). The subgradient projection method requires a quite low step size to avoid divergence. Here, the step size is adapted as in [12]. Using a sufficiently small step size, the method based on the subgradient method in some cases (user drops) requires a large number of iterations to find the solution.

Figure 4 depicts the number of iterations of the presented μ -SC method (46), (47) (outer loop), the inner loop and the bisection algorithm with the SDP. The number of required iterations increases linearly with the given precision. For a high precision, the presented algorithm requires more than 50 iterations. However, for a low precision, the presented solution with the μ -SC method (46) requires only a few iterations. For a low precision, the solution of the bisection algorithm with the SDP is not optimal (see Figure 2). The bisection algorithm requires a precision of $\epsilon = 10^{-5}$ to achieve a balanced SINR and the μ -SC method achieves the same result at a precision of 10^{-3} (Figure 1). The inner

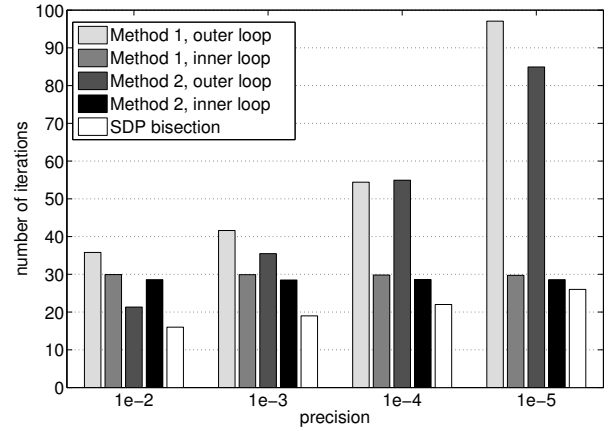


Fig. 4: Number of iterations for different precisions.

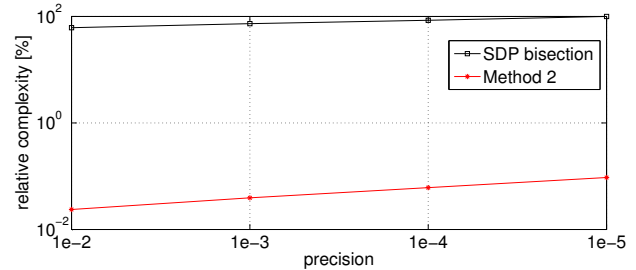


Fig. 5: Relative approximated complexity for given precision. The black curve denotes the relative approximated complexity of the bisection method with a SDP according to Section III-A and the blue curve shows the relative approximated complexity of the new iterative method of Section IV with updated (46). 100% is the number of flops of the bisection method with a SDP according to Section III-A at a precision of 10^{-5} .

loop has a precision of 10^{-5} , then it requires $K_I = 28$ iterations to converge. Regarding the complexity, the largest term of the \mathcal{O} -notation of the conventional convex solver based algorithm grows in $N^{3.5}N_A^{6.5}$ per iteration. The largest term of the new algorithm grows in $N^2N_A^3$ per iteration. Figure 5 depicts the inner part $g(N, N_A, K_O, K_I, \epsilon)$ of the \mathcal{O} -notation $\mathcal{O}(g(N, N_A, K_O, K_I, \epsilon))$ as a function of different precision values. It is evident that, the iterative method of Section IV with update (46) has less complexity compared to the SDP based bisection algorithm.

VI. CONCLUSION

This paper presents a new framework for uplink-downlink duality for the max-min beamforming problem with general power constraints. If an equivalent quasi-convex form of the max-min beamforming problem exists, e.g. if instantaneous CSI is available, strong duality is directly proved by a duality theorem for quasi-convex programming [22]. If long-term CSI in the form of higher rank spatial correlation matrices is given no equivalent quasi-convex form was known. However, in this case the max-min beamforming problem can be also solved by derived framework for uplink-downlink duality.

The presented dual problem is quasi-convex; hence it can be efficiently solved. Based on this framework, a low complexity iterative algorithm is presented. It is based on an inner and

outer loop and outperforms the known convex solver based solution.

APPENDIX A PROOF OF LEMMA 1

Proof: The derivation of the Lagrangian dual (25), (26) is a simple extension of [28]. The proof of the strong duality is an extension of the Lemma 2 given in [4]. With $\omega_i = \sqrt{p_i} \mathbf{v}_i$, the definition of the

$$\gamma^D = \max_{\Omega} \min_{i \in \mathcal{S}} \gamma_i^D(\Omega) \quad (48)$$

$$\text{s.t.} \quad \sum_{i \in \mathcal{U}} p_i \text{Tr}\{\mathbf{M}_i \mathbf{v}_i \mathbf{v}_i^H\} = P. \quad (49)$$

It has to be shown that, for the same sum power, the same SINR feasible region is the result. As in [4] with $\mathbf{D} = \text{diag}(\frac{1}{\mathbf{v}_1^H \mathbf{R}_{1,1} \mathbf{v}_1}, \dots, \frac{1}{\mathbf{v}_M^H \mathbf{R}_{M,M} \mathbf{v}_M})$, $\mathbf{P} = [p_1, \dots, p_M]$, $\boldsymbol{\lambda} = [\lambda_1, \dots, \lambda_M]$,

$$[\Psi]_{i,k} = \begin{cases} \mathbf{v}_k^H \mathbf{R}_{k,i} \mathbf{v}_k & k \neq i \\ 0 & k = i, \end{cases} \quad (50)$$

$\mathbf{d}(\mathbf{V}, \boldsymbol{\mu}) = [\text{Tr}\{\mathbf{M}_1 \mathbf{v}_1 \mathbf{v}_1^H\}, \dots, \text{Tr}\{\mathbf{M}_M \mathbf{v}_M \mathbf{v}_M^H\}]^T$, and for a balanced DL SINR $\gamma^D = \gamma$, the following set of equations for fixed \mathbf{V} and \mathbf{M}_i is the result:

$$\frac{1}{\gamma} \mathbf{p} = \mathbf{D} \Psi \mathbf{p} + \mathbf{D} \cdot \mathbf{1} \Leftrightarrow \mathbf{p} = (\frac{1}{\gamma} \mathbf{D}^{-1} - \Psi)^{-1} \mathbf{1}. \quad (51)$$

The same holds for the balanced UL SINR $\gamma^U = \gamma$:

$$\begin{aligned} \frac{1}{\gamma} \boldsymbol{\lambda} &= \mathbf{D} \Psi^T \boldsymbol{\lambda} + \mathbf{D} \cdot \mathbf{d}(\mathbf{V}, \boldsymbol{\mu}) \\ \Leftrightarrow \boldsymbol{\lambda} &= (\frac{1}{\gamma} \mathbf{D}^{-1} - \Psi^T)^{-1} \mathbf{d}(\mathbf{V}, \boldsymbol{\mu}). \end{aligned} \quad (52)$$

Note, $\rho(\gamma \mathbf{D} \Psi) < 1$, because the MBP is always feasible (23). The question is now whether for both problems (23) and (25), in the weighted sum power case the same sum power for the feasible SINR γ will be the result. With (52), the UL power of the UL MBP is given by:

$$\begin{aligned} \boldsymbol{\lambda}^T \cdot \mathbf{1} &= [(\frac{1}{\gamma} \mathbf{D}^{-1} - \Psi^T)^{-1} \mathbf{d}(\mathbf{V}, \boldsymbol{\mu})]^T \mathbf{1} \\ &= \mathbf{d}^T(\mathbf{V}, \{\mathbf{M}_i\}) \underbrace{(\frac{1}{\gamma} \mathbf{D}^{-1} - \Psi)^{-1} \mathbf{1}}_{=\mathbf{p}} = \mathbf{d}^T(\mathbf{V}, \boldsymbol{\mu}) \mathbf{p}. \end{aligned} \quad (53)$$

The last line is the weighted sum power constraint (49) of the DL MBP. Thus, the UL and DL MBP achieve the same SINR with the same total transmit power. Therefore, both problems have the same balanced SINR. ■

APPENDIX B

PROOF OF THE LAGRANGIAN DUAL IN PROPOSITION 6

Proof: Using the Lagrangian of the primal problem (10)

$$\begin{aligned} L(\gamma, \boldsymbol{\Omega}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= \gamma + \sum_{i \in \mathcal{S}} \text{Tr}\{\mathbf{M}_i \mathbf{P}_i\} - \sum_{i \in \mathcal{U}} \lambda_i \\ &\quad + \sum_{i \in \mathcal{U}} \omega_i^H \left[\frac{\lambda_i}{\gamma} \mathbf{R}_{i,i} - \mathbf{M}_i - \sum_{l \in \mathcal{U}, l \neq i} \lambda_l \mathbf{R}_{i,l} \right] \omega_i, \end{aligned}$$

the dual function of this problem is:

$$l(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sup_{\gamma, \boldsymbol{\Omega}} L(\gamma, \boldsymbol{\Omega}, \boldsymbol{\lambda}, \boldsymbol{\mu}).$$

By observation, it can be inferred that $g(\boldsymbol{\lambda}, \boldsymbol{\mu}) < \infty$ only if

$$\frac{\lambda_i}{\gamma} \mathbf{R}_{i,i} - \mathbf{M}_i - \sum_{l \in \mathcal{U}, l \neq i} \lambda_l \mathbf{R}_{i,l} \preceq 0, \quad (54)$$

from which the constraint of the dual function follows. Hence, the Lagrangian dual problem can be stated as:

$$\gamma^L = \min_{\boldsymbol{\lambda}, \boldsymbol{\mu}} \max_{\gamma} \gamma + \sum_{i \in \mathcal{U}} \text{Tr}\{\mathbf{M}_i \mathbf{P}_i\} - \sum_{i \in \mathcal{S}} \lambda_i \quad (55)$$

$$\text{s.t.} \quad \mathbf{M}_i + \sum_{\substack{l \in \mathcal{U} \\ l \neq i}} \lambda_l \mathbf{R}_{i,l} \succeq \frac{\lambda_i}{\gamma} \mathbf{R}_{i,i},$$

$$\mathbf{M}_i \succeq 0, \lambda_i \geq 0, \forall i \in \mathcal{U}.$$

With the definition of the new optimization variable χ and the additional constraint $\chi \geq \sum_{i \in \mathcal{U}} \lambda_i$, the problem can be rephrased to

$$\gamma^L = \max_{\chi} \min_{\boldsymbol{\lambda}, \boldsymbol{\mu}} \max_{\gamma} \gamma + \sum_{i \in \mathcal{S}} \text{Tr}\{\mathbf{M}_i \mathbf{P}_i\} - \chi \quad (56)$$

$$\text{s.t.} \quad \mathbf{M}_i + \sum_{\substack{l \in \mathcal{U} \\ l \neq i}} \lambda_l \mathbf{R}_{i,l} \succeq \frac{\lambda_i}{\gamma} \mathbf{R}_{i,i},$$

$$\chi \geq \sum_{i \in \mathcal{U}} \lambda_i,$$

$$\mathbf{M}_i \succeq 0, \lambda_i \geq 0, \forall i \in \mathcal{U}.$$

Using the additional variable substitutions $\chi = \chi' P$, and $\mathbf{M}_i = \mathbf{M}'_i \chi'$, where $P = \sum_{i \in \mathcal{U}} \text{Tr}\{\mathbf{M}'_i \mathbf{P}_i\}$ and $\lambda_i = \lambda'_i \chi'$, the following simplification of the problem (56) is given by:

$$\gamma^L = \min_{\boldsymbol{\lambda}', \boldsymbol{\mu}'} \max_{\gamma} \gamma \quad (57)$$

$$\text{s.t.} \quad \mathbf{M}'_i + \sum_{\substack{l \in \mathcal{U} \\ l \neq i}} \lambda'_l \mathbf{R}_{i,l} \succeq \frac{\lambda'_i}{\gamma} \mathbf{R}_{i,i},$$

$$\sum_{i \in \mathcal{U}} \lambda'_i \leq \sum_{i \in \mathcal{U}} \text{Tr}\{\mathbf{M}'_i \mathbf{P}_i\},$$

$$\mathbf{M}'_i \succeq 0, \lambda'_i \geq 0, \forall i \in \mathcal{U}.$$

With the substitutions $\lambda_i = \lambda'_i$ and $\mathbf{M}_i = \mathbf{M}'_i$ and multiplying both sides of the first constraint in (57) by \mathbf{v}_i^H from the left and \mathbf{v}_i from the right [37], it can be rewritten as

$$\gamma \geq \frac{\lambda_i \mathbf{v}_i^H \mathbf{R}_{i,i} \mathbf{v}_i}{\mathbf{v}_i^H (\mathbf{M}_i + \sum_{\substack{l \in \mathcal{U} \\ l \neq i}} \lambda_l \mathbf{R}_{i,l}) \mathbf{v}_i}. \quad (58)$$

With the assumption of a balanced SINR among all users (according to Definition 2), the first constraint is met with equality if the \mathbf{M}_i s and \mathbf{V} are fixed [12]. Therefore, the reversal of the SINR constraints and the reversal of the minimization as a maximization over λ_i do not affect the

optimal solution [12].

$$\begin{aligned} \gamma^L &= \min_{\boldsymbol{\mu}} \max_{\boldsymbol{\lambda}, \boldsymbol{\gamma}, \mathbf{V}} \gamma & (59) \\ \text{s.t. } & \gamma \leq \gamma_i^U(\boldsymbol{\mu}, \boldsymbol{\lambda}, \mathbf{v}_i) \\ & \mathbf{M}_i \succeq 0, \lambda_i \geq 0, \forall i \in \mathcal{U}, \\ & \sum_{i \in \mathcal{U}} \lambda_i \leq \sum_{i \in \mathcal{U}} \text{Tr}\{\mathbf{M}_i \mathbf{P}_i\}. \end{aligned}$$

Replacing γ with the right hand side of the first constraints in (59), the optimization problem is formulated as in (39). ■

APPENDIX C COMPLEXITY ANALYSIS

Let N be the number of users and N_A be the number of antennas per array, as in [28], the complexity of the inner loop can be summarized as follows:

- The complexity of determining the matrix $\boldsymbol{\Sigma}_i$ (40) is in the order of $\mathcal{O}(NN_A^2)$ [28].
- The computation of the inverse matrix $\boldsymbol{\Sigma}_i^{-1}$ is in the order of $\mathcal{O}(N_A^3)$.
- The matrix–matrix multiplication of the symmetric matrix $\boldsymbol{\Sigma}_i^{-1} \mathbf{R}_{i,i}$ has a complexity in the order of $\mathcal{O}(N_A^3)$.
- The matrix–vector multiplication $\boldsymbol{\Sigma}_i^{-1} \mathbf{R}_{i,i} \mathbf{v}_i$ has a complexity in the order of $\mathcal{O}(N_A^2)$.
- The normalization step $\|\mathbf{v}_i\| = 1$ has a low complexity compared to the other steps, therefore, it is ignored.
- The eigenvalue computation (43) consists of two vector–matrix–vector products. The complexity is in the order of $\mathcal{O}(N_A^2)$.

Consequently, the order of the total complexity of all these operations can be upper bounded by $\mathcal{O}(NN_A^3 + NN_A^2)$. These steps are made for each of the N users; hence, the total complexity is in the order of $\mathcal{O}(N^2N_A^3 + N^2N_A^2)$. Based on similar simple investigations, the order of the downlink power computation and the scaling step (46) can be upper bounded by $\mathcal{O}(N^3N_A^2)$. Assuming the inner loop needs K_I iterations, the total complexity for one outer iteration is in the order of $\mathcal{O}(K_I N^2(N_A^3 + N_A^2) + N^3N_A^2)$. Assuming the outer loop needs K_O iterations, the order of the total complexity is then given by $\mathcal{O}(K_O K_I N^2(N_A^3 + N_A^2) + K_O N^3N_A^2)$. The SDP (12) requires interior point methods for solving it. The complexity of a fast interior point method [38] can be approximated by $\mathcal{O}(n^{3.5} \log(1/\epsilon))$, where n is the total variable size [39]. In [24], the authors estimate the order of the complexity of the convex solver based feasibility check problem to $\mathcal{O}(\log(1/\epsilon) \sqrt{NN_A} (N^3N_A^6 + N^2N_A^2))$. Assuming there are K_O outer iterations needed by the bisection algorithm, the total complexity is in the order of: $\mathcal{O}(K_O \log(1/\epsilon) \sqrt{NN_A} (N^3N_A^6 + N^2N_A^2))$.

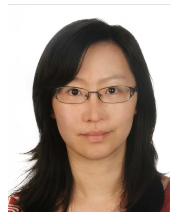
REFERENCES

- [1] M. Bengtsson and B. Ottersten, "Optimum and suboptimum transmit beamforming," in *Handbook of Antennas in Wireless Communications*. Boca Raton, USA: CRC Press, 2002.
- [2] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge University Press, 2004.
- [3] A. Tölli, M. Codreanu, and M. Juntti, "Linear multiuser MIMO transceiver design with quality of service and per-antenna power constraints," *IEEE Transactions on Signal Processing*, vol. 56, no. 7, pp. 3049–3055, Jul. 2008.
- [4] M. Schubert and H. Boche, "Solution of the multiuser downlink beamforming problem with individual SINR constraints," *IEEE Transactions on Vehicular Technology*, vol. 53, no. 1, pp. 18–28, Jan. 2004.
- [5] F. Rashid-Farrokhi, K. J. R. Liu, and L. Tassiulas, "Transmit beamforming and power control for cellular wireless systems," *IEEE Journal on Selected Areas in Communications*, vol. 16, no. 8, pp. 1437–1450, Oct. 1998.
- [6] W. Yang and G. Xu, "Optimal downlink power assignment for smart antenna systems," in *IEEE International Conference on Acoustics, Speech and Signal Processing, (ICASSP)*, vol. 6, Seattle, USA, May 1998, pp. 3337–3340.
- [7] H. Weingarten, Y. Steinberg, and S. Shamai, "The capacity region of the gaussian multiple-input multiple-output broadcast channel," *IEEE Transactions on Information Theory*, vol. 52, no. 9, pp. 3936–3964, Sep. 2006.
- [8] A. Wiesel, Y. C. Eldar, and S. Shamai, "Linear precoding via conic optimization for fixed MIMO receivers," *IEEE Transactions on Signal Processing*, vol. 54, no. 1, pp. 161–176, Jan. 2006.
- [9] D. W. H. Cai, T. Q. S. Quek, and C. W. Tan, "A unified analysis of max-min weighted SINR for MIMO downlink system," *IEEE Transactions on Signal Processing*, vol. 59, no. 8, pp. 3850–3862, Aug. 2011.
- [10] H. Boche and M. Schubert, "A general duality theory for uplink and downlink beamforming," in *56th IEEE Vehicular Technology Conference, (VTC Fall)*, vol. 1, Vancouver, Canada, Sep. 2002, pp. 87–91.
- [11] B. Song, R. L. Cruz, and B. D. Rao, "Network duality and its application to multi-user mimo wireless networks with sinr constraints," in *IEEE International Conference on Communications, (ICC)*, Seoul, Korea, May 2005, pp. 2684–2689.
- [12] W. Yu and T. Lan, "Transmitter optimization for the multi-antenna downlink with per-antenna power constraints," *IEEE Transactions on Signal Processing*, vol. 55, no. 6, pp. 2646–2660, Jun. 2007.
- [13] C. W. Tan, M. Chiang, and R. Srikant, "Maximizing sum rate and minimizing MSE on multiuser downlink: Optimality, fast algorithms and equivalence via max-min SIR," in *IEEE International Symposium on Information Theory, (ISIT)*, Seoul, Korea, Jul. 2009.
- [14] —, "Maximizing sum rate and minimizing MSE on multiuser downlink: Optimality, fast algorithms and equivalence via max-min SINR," *IEEE Transactions on Signal Processing*, vol. 59, no. 12, pp. 6127–6143, Dec. 2011.
- [15] —, "Fast algorithms and performance bounds for sum rate maximization in wireless networks," in *IEEE International Conference on Computer Communications (INFOCOM)*, Rio de Janeiro, Brazil, Apr. 2009, pp. 1350–1358.
- [16] D. W. H. Cai, T. Q. S. Quek, C. W. Tan, and S. H. Low, "Max-min weighted SINR in coordinated multicell MIMO downlink," in *IEEE International Symposium on Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt)*, Princeton, USA, May 2011, pp. 286–293.
- [17] D. W. H. Cai, T. Q. S. Quek, and C. W. Tan, "Max-min SINR coordinated multipoint downlink transmission-duality and algorithms," *IEEE Transactions on Signal Processing*, vol. 60, no. 10, pp. 5384 – 5395, Oct. 2012.
- [18] F. Negro, M. Cardone, I. Ghauri, and D. T. M. Slock, "SINR balancing and beamforming for the MISO interference channel," in *22nd IEEE International Symposium on Personal, Indoor and Mobile Radio Communications (PIMRC)*, Toronto, Canada, Sep. 2011, pp. 1552–1556.
- [19] H. Dahrouj and W. Yu, "Coordinated beamforming for the multicell multi-antenna wireless system," *IEEE Transactions on Wireless Communications*, vol. 9, no. 5, pp. 1748–1759, May 2010.
- [20] G. Dartmann, W. Afzal, X. Gong, and G. Ascheid, "Low complexity cooperative downlink beamforming in multiuser multicell networks," in *12th IEEE International Conference on Communication Technology, (ICCT)*, Nanjing, China, Nov. 2010.
- [21] G. Dartmann, X. Gong, and G. Ascheid, "Max-min beamforming with cooperative multipoint transmission," in *74th IEEE Vehicular Technology Conference, (VTC Fall)*, San Francisco, USA, Sep. 2011.
- [22] D. G. Luenberger, "Quasi-convex programming," *SIAM Journal of Applied Mathematics*, vol. 16, no. 5, pp. 1090–1095, Sep. 1968.
- [23] P. Viswanath, D. N. C. Tse, and R. Laroia, "Opportunistic beamforming using dumb antennas," *IEEE Transactions on Information Theory*, vol. 48, no. 6, pp. 1277–1294, Jun. 2002.
- [24] E. Karipidis, N. D. Sidiropoulos, and Z.-Q. Luo, "Quality of service and max-min fair transmit beamforming to multiple cochannel multicast

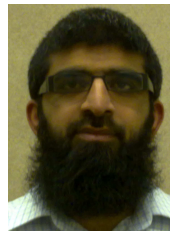
- groups," *IEEE Transactions on Signal Processing*, vol. 56, no. 3, pp. 1268–1279, 2008.
- [25] H. J. Greenberg and W. P. Pierskalla, "Surrogate mathematical programming," *Operations Research*, vol. 18, no. 5, pp. 924–93, Sep. 1970.
- [26] T. Nieuwenhuizen, "Johri's general dual, the lagrangian dual, and the surrogate dual," *European Journal of Operational Research*, vol. 117, no. 1, pp. 183–196, Aug. 1999.
- [27] D. Li and X. Sun, "Nonlinear integer programming," in *International Series in Operations Research & Management Science*. Springer, 2006, vol. 84.
- [28] X. Gong, M. Jordan, G. Dartmann, and G. Ascheid, "Max-min beamforming for multicell downlink systems using long-term channel statistics," in *20th IEEE International Symposium on Personal, Indoor and Mobile Radio Communications (PIMRC)*, Tokyo, Japan, Sep. 2009, pp. 803–807.
- [29] R. Zhang and S. Cui, "Cooperative interference management with MISO beamforming," *IEEE Transactions on Signal Processing*, vol. 58, no. 10, pp. 5450–5458, Oct. 2010.
- [30] M. Schubert and H. Boche, "QoS-based resource allocation and transceiver optimization," *Foundations and Trends in Communications and Information Theory*, vol. 2, no. 6, 2005.
- [31] M. Schubert, N. Vucic, and H. Boche, "SIR balancing for strongly connected interference networks-existence and uniqueness of a solution," in *7th IEEE International Symposium on Wireless Communication Systems (ISWCS)*, York, UK, Sep. 2010, pp. 184–188.
- [32] G. H. Golub and C. F. van Loan, *Matrix computations*. The Johns Hopkins University Press, 1996.
- [33] U. Krause, "Concave perron-frobenius theory and applications," *Nonlinear Analysis, (TMA)*, vol. 47, pp. 1457–1466, Aug. 2004.
- [34] M. Jordan, M. Senst, G. Ascheid, and H. Meyr, "Long-term beamforming in single frequency networks using semidefinite relaxation," in *67th IEEE Vehicular Technology Conference, (VTC Spring)*, Marina Bay, Singapore, May 2008, pp. 275–279.
- [35] K. I. Pedersen, P. E. Mogensen, and B. H. Fleury, "Spatial channel characteristics in outdoor environments and their impact on BS antenna system performance," in *48th IEEE Vehicular Technology Conference, (VTC 98)*, vol. 2, Ottawa, Canada, May 1998, pp. 719–723.
- [36] J. F. Sturm. (2009) Using sedumi 1.21, a matlab toolbox for optimization over symmetric cones. [Online]. Available: <http://sedumi.ie.lehigh.edu/>
- [37] J. Yang and D. K. Kim, "Multi-cell uplink-downlink beamforming throughput duality based on Lagrangian duality with per-base station power constraints," *IEEE Communications Letters*, vol. 12, no. 4, pp. 277–279, Apr. 2008.
- [38] C. Helmberg, F. Rendl, R. Vanderbei, and H. Wolkowicz, "An interior-point method for semidefinite programming," *SIAM Journal on Optimization*, vol. 6, no. 2, pp. 342–361, May 1996.
- [39] Z.-Q. Luo, W.-K. Ma, A. M.-C. So, Y. Ye, and S. Zhang, "Semidefinite relaxation of quadratic optimization problems," *IEEE Signal Processing Magazine, IEEE*, vol. 27, no. 3, pp. 20–34, May 2010.



Guido Dartmann (S'11) received the Diploma degree in computer engineering from RWTH Aachen University, Germany, in 2007. He finished his Diploma thesis in the field of polyphase filterbank design and he is currently working as a research assistant and pursuing the PhD degree at the Chair for Integrated Signal Processing Systems, RWTH Aachen University. His main areas of research interests are interference mitigation, cooperative communication, beamforming, and filterbank based multicarrier systems.



Xitao Gong (S'09) received the Bachelor and Master degrees from Southeast University, China, in 2004 and 2007 respectively. She is currently working as a research assistant and pursuing the PhD degree at the Chair for Integrated Signal Processing Systems, RWTH Aachen University. Her main areas of research interest are interference detection and mitigation in cognitive and cooperative networks.



Waqas Afzal received his B.Sc. degree in Communication Systems Engineering from the Institute of Space Technology (IST), Islamabad, Pakistan, in 2007 and his M.S. in Communications Engineering from RWTH Aachen University, Aachen, Germany, in 2010. Since Feb. 2012 he is working towards his PhD in Electrical Engineering at the King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia. His current research interests include motion planning, robotics, signal processing in wireless communication and application of optimization techniques in wireless communication.



Gerd Ascheid (SM08) received his Diploma and PhD degrees in Electrical Engineering (Communications Eng.) from RWTH Aachen University. In 1988 he started as a co-founder CADIS GmbH which successfully brought the system simulation tool COS-SAP to the market. From 1994-2003 Gerd Ascheid was Director / Senior Director with Synopsys, a California-based EDA market leader. In 2002 he was co-founder of LisaTek whose processor design tools are now part of the Synopsys product portfolio. Since April 2003 Gerd Ascheid heads the Institute for Integrated Signal Processing Systems of RWTH Aachen University. He is also coordinator of the UMIC (Ultra-high speed Mobile Information and Communication) Research Centre at RWTH Aachen University. His research interest is in wireless communication algorithms and application specific integrated platforms, in particular, for mobile terminals.